Soln's Final Exam 167 F08

Problem #1: (a) Let $\langle x,y\rangle$ denote the complex inner product

$$\langle x,y\rangle = \bar{x}^T y = \bar{x}\cdot y$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{matrix} & \text{dot} \end{array}$$

Then $A=A^T$ real $\Rightarrow \langle Ax,y\rangle = \bar{x}^T\bar{A}^T y = \bar{x}^T A y$

$$= \underset{\text{dot}}{\bar{x}\cdot Ay} = \langle x, Ay\rangle$$

Thus if $Ax=\lambda x$, $x\neq 0$, λ complex, then

$\underset{\text{dot}}{\bar{\lambda}\bar{x}\cdot x} = \langle Ax,x\rangle = \langle x,Ax\rangle = \bar{x}\cdot\lambda x$

$$\Rightarrow \bar{\lambda}|x|^2 = \lambda|x|^2$$

$$\Rightarrow \bar{\lambda}=\lambda \text{ since } x\neq 0 \Rightarrow \lambda \text{ real} \checkmark$$

(b) $Ax_1=\lambda_1 x_1$ & $Ax_2=\lambda_2 x_2 \Rightarrow$

$\lambda_1 x_1\cdot x_2 = \langle Ax_1, x_2\rangle = \langle x_1, Ax_2\rangle = \langle x_1, \lambda_2 x_2\rangle = \lambda_2 x_1\cdot x_2$

$\therefore$ if $\lambda_1\neq\lambda_2$ we must have $\langle x_1,x_2\rangle = 0 \checkmark$

Problem 2 Let $A = \begin{bmatrix} 1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{bmatrix}$

(a) Find evals of A

(b) Find an O.N. basis of e-vectors of A

(c) Find $\lim_{n \to \infty} A^n$

Soln (a) $\left(\frac{1}{2} - \lambda\right)(-\lambda) - \frac{1}{2} = 0 \iff \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0$

$\iff (\lambda + 1/2)(\lambda - 1) = 0 \quad \lambda_1 = -\frac{1}{2}, \lambda_2 = 1$

(b) $\lambda_1 = -\frac{1}{2} \quad \begin{bmatrix} 1/2 + 1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \quad \begin{matrix} a + \frac{1}{\sqrt{2}} b = 0 \\ \frac{1}{\sqrt{2}} a + \frac{1}{2} b = 0 \end{matrix} \quad \begin{matrix} b = -\sqrt{2} a \\ a = 1 \\ b = -\sqrt{2} \end{matrix}$

$\hat{R}_1 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \quad \|\hat{R}_1\| = \sqrt{1 + 2} = \sqrt{3}$

$R_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$

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$\lambda_2 = 1 \quad \begin{bmatrix} \frac{1}{2}-1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$

$-\frac{1}{2}a + \frac{1}{\sqrt{2}}b = 0 \quad a = \sqrt{2}b$

$\frac{1}{\sqrt{2}}a - b = 0 \quad a = \sqrt{2}b$

$b = 1,\ a = \sqrt{2}$

$\hat{R}_2 = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} \quad \|\hat{R}_2\| = \sqrt{3}$

$R_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$

ON Basis: $\left\{ \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} \right\}$

(d) $S = \begin{bmatrix} | & | \\ R_1 & R_2 \\ | & | \end{bmatrix} \quad S^{-1} = S^T$ &

$A = S\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}S^T \Rightarrow A^n = S\begin{bmatrix} (-\frac{1}{2})^n & 0 \\ 0 & 1^n \end{bmatrix}S^T$

$\therefore A^n \longrightarrow S\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}S^T = \begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \cdot \frac{1}{3}$

$= \frac{1}{3}\begin{bmatrix} 0 & \sqrt{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$ ✓

Problem #3: Assume $Av_i = Bv_i$ for basis $\{v_1, \dots, v_n\}$ of $\mathbb{R}^n$. Then for $x \in \mathbb{R}^n$, we have

$$x = c_1 v_1 + \cdots + c_n v_n$$

so

$$Ax = A(c_1 v_1 + \cdots + c_n v_n) = c_1 A v_1 + \cdots + c_n A v_n$$
$$= c_1 B v_1 + \cdots + c_n B v_n$$
$$= B(c_1 v_1 + \cdots + c_n v_n) = Bx$$

Thus in particular, letting $e_i = (0 \cdots \underset{\substack{\uparrow \\ \text{ith slot}}}{1} \cdots 0)$

and setting $A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$, $B = \begin{bmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{bmatrix}$ we have

$a_i = Ae_i = Be_i = b_i$ so each ith colm of A equals the ith colm of B $\Rightarrow$ $A = B$ ✓

Problem #4: Let $x_1 = y_1 - a$, $x_2 = y_2 - b$ so $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ satisfy

$$\underbrace{\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}}_{x'} = \underbrace{\begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$$

(a) If $y_1 = a$, $y_2 = b$ then $\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = 0 = \begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} a-a \\ b-b \end{bmatrix}$$ ✓

(b) We have $y(t) = x(t) + \begin{bmatrix} a \\ b \end{bmatrix}$, so to show $y(t) \to \begin{bmatrix} a \\ b \end{bmatrix}$ it suffices to show $x(t) \to 0$.

• Evals: $|A - \lambda I| = \begin{vmatrix} -4-\lambda & 3 \\ -2 & 1-\lambda \end{vmatrix} = -(4+\lambda)(1-\lambda) + 6$

$$= \lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda = -1, -2$$

$\lambda_1 = -1$ $\quad \begin{bmatrix} -4+1 & 3 \\ -2 & 1+1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \quad -a+b=0$

$R_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad a=1, b=1$

$\lambda_2 = -2$ $\quad \begin{bmatrix} -4+2 & 3 \\ -2 & 1+2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \quad -2a+3b=0$

$R_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\quad a=3, b=2$

$$x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \xrightarrow{t \to \infty} 0$$

In particular

$$y(t) = \begin{bmatrix} a \\ b \end{bmatrix} + c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \xrightarrow{t \to \infty} \begin{bmatrix} a \\ b \end{bmatrix}$$

#5 (a) Assume $x^T A^T A x > 0 \ \forall x \in \mathbb{R}^n$. Now $A^T A$ symmetric $\Rightarrow$ it has real evals and an ON basis of eigenvectors. Thus $\exists$ n pairs (λ_i, v_i) st $A^T A v_i = \lambda_i v_i$ and $\{v_1, \dots, v_n\}$ an ON basis. It remains to show $\lambda_i > 0$.

But $v_i^T A^T A v_i = v_i^T \lambda_i v_i = \lambda |v_i|^2 > 0$

so we must have $\lambda_i > 0$ ✓

(b) $A^T A v_i = \lambda_i v_i$ so mult by A to get

$(AA^T) A v_i = \lambda_i A v_i \Rightarrow (\lambda_i, Av_i)$ are eigen-prs for AA^T. Moreover, setting $u_i = Av_i$,

$\langle u_i, u_j \rangle = \langle Av_i, Av_j \rangle = v_i^T A^T A v_j = v_i^T \lambda_i v_j$

$= \lambda_i \langle v_i, v_j \rangle = 0 \quad i \neq j \Rightarrow$

$\{u_i\}$ are orthogonal.

Problem #6: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \; ; \; A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \ldots A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

$$e^A = \sum_{n=0}^{\infty} \frac{\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}}{n!} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{1}{n!} & \sum_{n=0}^{\infty} \frac{n}{n!} \\ 0 & \sum_{n=0}^{\infty} \frac{1}{n!} \end{bmatrix}$$

Now $\sum_{n=0}^{\infty} \frac{1}{n!} = e^1 = e$

& $\sum_{n=0}^{\infty} \frac{n}{n!} = 1 + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} = \sum_{n=0}^{\infty} \frac{1}{n!} = e$

$$\Rightarrow e^A = \begin{bmatrix} e & e \\ 0 & e \end{bmatrix}.$$

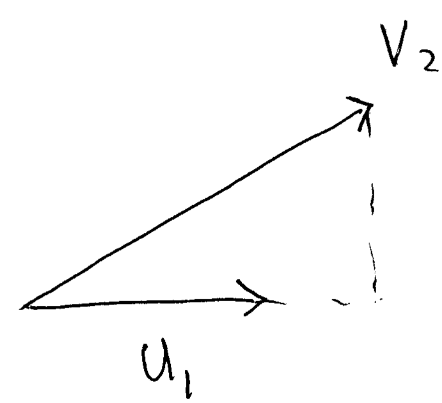
#7 (a) $x = c_1 u_1 + \cdots + c_n u_n$, $\{u_i\}$ o.n. basis $\Rightarrow$

$$u_i \cdot x = c_1 u_i \cdot u_1 + \cdots + c_n u_i \cdot u_n = c_i u_i \cdot u_i = c_i$$

$$\therefore \quad c_i = (u_i \cdot x)$$

(b) $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$



$$u_2 = \frac{v_2 - (u_1 \cdot v_2) u_1}{\| \quad " \quad \|}$$

$$(u_1 \cdot v_2) u_1 = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$v_2 - (u_1 \cdot v_2) u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\| v_2 - (u_1 \cdot v_2) u_1 \| = \frac{\sqrt{2}}{2}.$$

$$u_2 = \frac{1}{2} \cdot \frac{2}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$u_3 = \frac{v_3 - (u_1 \cdot v_3)u_1 - (u_2 \cdot v_3)u_2}{\| \quad \text{''} \quad \|}$$

$$(u_1 \cdot v_3)u_1 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$(u_2 \cdot v_3)u_2 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

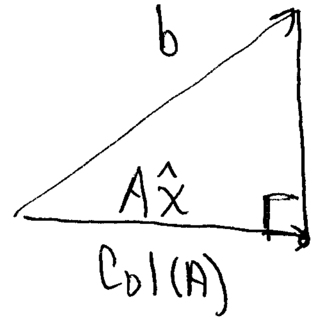
$$v_3 - (u_1 \cdot v_3)u_1 - (u_1 \cdot v_3)u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \; u_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Conclude: $u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $u_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Problem #8 :



(a) $A\hat{x} \in Col(A)$ is <u>closest</u> to b

when $b - A\hat{x} \perp Col(A) \iff$

$$A^T(b - A\hat{x}) = 0$$

$$\iff A^T b - A^T A\hat{x} = 0$$

$$\iff A^T A\hat{x} = A^T b$$

Now $\text{rank}(A) = n \Rightarrow$ rank of $\underset{n\times n}{A^T A} = n \Rightarrow (A^T A)$

has an inverse $\Rightarrow$

$$\hat{x} = (A^T A)^{-1} A^T b$$

Problem #8

(b) Find line $b = C + Dt$ that best fits data points

$$(t_1, b_1) = (1, -1)\ ;\ (t_2, b_2) = (-1, 1)\ ;\ (t_3, b_3) = (1, 2)$$

Let $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \end{bmatrix} = \begin{bmatrix} 1 & +1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$

Want: $Ax =$ but settle for $A\hat{x} = b$ $\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix}$

By (a) $\hat{x} = (A^TA)^{-1}A^Tb$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \end{bmatrix} = \begin{bmatrix} 3 & \Sigma t_i \\ \Sigma t_i & \Sigma t_i^2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{8} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1}A^Tb = \frac{1}{8} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ -4 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} C \\ D \end{bmatrix}$$