Name:

Student ID:
Section:

1	2	3	4	5	6	Total

MIDTERM EXAM
Math 167
Temple-F08

Problem 1. (20pts) True or False:
(Here A is an arbitrary $m \times n$ matrix, $Col(A) = C(A)$ denotes the Column Space of A, and $Row(A)$ the Row Space of A.)

(a) $C(A^T) = Row(A)$.

(b) $dim\{Row(A)\} = dim\{Col(A)\}$.

(c) For every $b \in Col(A)$ there exists an $\mathbf{x} \in Row(A)$ such that $A\mathbf{x} = \mathbf{b}$.

(d) For every $b \in Col(A)$ there exists *no more than one* $\mathbf{x} \in Row(A)$ such that $A\mathbf{x} = \mathbf{b}$.

(e) A^{-1} exists if and only if $det(A) = 0$.

(f) The number of operations required to do back substitution is order n^2.

(g) The number of operations required to compute L and U via Gaussian Elimination is order n^3.

(h) Every invertible $n \times n$ matrix A has an LDU decomposition. (L=lower triangular, 1's on diagonal, U=upper triangular, 1's on diagonal, D=diagonal.)

(i) If A is an $n \times n$ invertible matrix, then $A^T A = AA^T$.

(j) If $A = A^T$ and $A = LDU$ is an *LDU-decomposition* of A, then $L = U^T$.

Problem 2. (15pts) Let $\{v_1, ..., v_n\}$ be a basis, (i.e., a linearly independent spanning set), for a vector space V. Prove that every element $v \in V$ can be expressed *uniquely* as a linear combination of $v_1, ..., v_n$.

Problem 3. (15pts) Let A be an $m \times n$ matrix viewed as a transformation taking $\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n \in \mathcal{R}^n$ to $A\mathbf{x} = \mathbf{b} \in \mathbf{R}^m$. Label $\mathbf{x}_r, \mathbf{x}_n, \mathbf{b}, Row(A), Col(A), N(A), N(A^T), \mathcal{R}^n$, and $\mathcal{R}^m$ in the diagram below: ($\mathbf{x}_r \in Row(A)$, $\mathbf{x}_n \perp \mathbf{x}_r$.)

Problem 4. (17pts) Let

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix},$$

and define $A = LU$.

(a) Describe the most efficient way to solve $A\mathbf{x} = \mathbf{b}$.

(b) Solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$.

Problem 5. (15pts) Let $A = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & 3 & 0 & 1 \\ -2 & 1 & 1 & 0 \end{bmatrix}$.

(a) Find the *pivots* and determine the *free variables.*

(b) Find three elementary matrices E_k of form $E_{ij}(a)$ s.t.

$$E_3 \cdot E_2 \cdot E_1 \cdot A = U,$$

where U is upper triangular.

(c) Find a 3×3 lower triangular matrix L with 1's on the diagonal such that $A = LU$.

Problem 6. **(18pts)** Let $A_{m \times n}$ satisfy $m > n$, and assume the columns of A form a basis for R^n. For $b \in \mathcal{R}^m$, let $A\hat{x}$ denote the element of $Col(A)$ closest to b.

(a) Show that $\hat{x} = (A^T A)^{-1} A^T \mathbf{b}$.

(b) Find the line $b = C + Dt$ that best fits the data points

$$(t_1, b_1) = (1, -1); \quad (t_2, b_2) = (-1, 1); \quad (t_3, b_3) = (1, 2)$$

in the sense that $\sum_{i=1}^{3} \{b_i - (C + Dt_i)\}^2$ is minimized.