Solutions

Name:

Student ID:
Section:

1	2	3	4	5	6	Total

MIDTERM EXAM
Math 167
Temple-F08

Problem 1. (20pts) True or False:
(Here A is an arbitrary $m \times n$ matrix, $Col(A) = C(A)$ denotes the Column Space of A, and $Row(A)$ the Row Space of A.)

T **(a)** $C(A^T)= Row(A)$.

T **(b)** $dim\{Row(A)\} = dim\{Col(A)\}$.

T **(c)** For every $b \in Col(A)$ there exists an $\mathbf{x} \in Row(A)$ such that $A\mathbf{x} = \mathbf{b}$.

T **(d)** For every $b \in Col(A)$ there exists *no more than one* $\mathbf{x} \in Row(A)$ such that $A\mathbf{x} = \mathbf{b}$.

F **(e)** A^{-1} exists if and only if $det(A) = 0$.

T **(f)** The number of operations required to do back substitution is order n^2.

T **(g)** The number of operations required to compute L and U via Gaussian Elimination is order n^3.

F **(h)** Every invertible $n \times n$ matrix A has an LDU decomposition. (L=lower triangular, 1's on diagonal, U=upper triangular, 1's on diagonal, D=diagonal.)

F **(i)** If A is an $n \times n$ invertible matrix, then $A^TA = AA^T$.

T **(j)** If $A = A^T$ and $A = LDU$ is an *LDU-decomposition* of A, then $L = U^T$.

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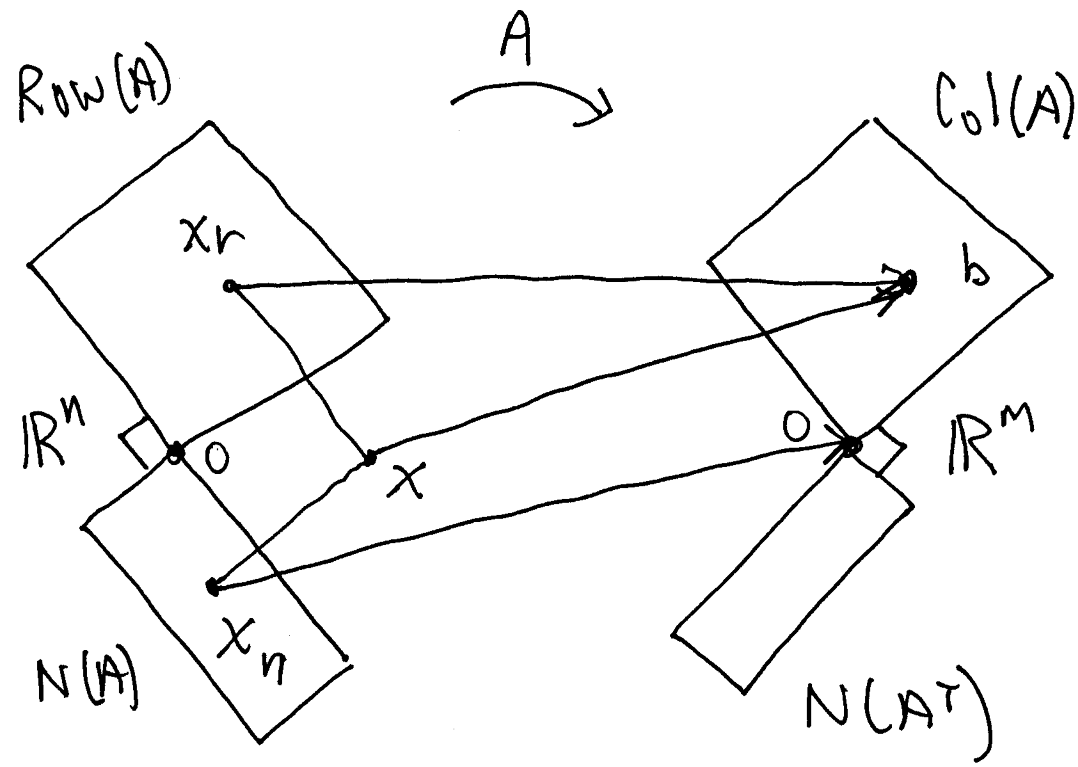
Midterm Exam Solutions

(#2) Since $\{v_1, \dots, v_n\}$ span V, for each $v \in V$ $\exists$ $c_1, \dots, c_n$ st $v = c_1 v_1 + \cdots + c_n v_n$. If $c_1 v_1 + \cdots + c_n v_n = c_1' v_1 + \cdots + c_n' v_n$ then

$$(c_1 - c_1') v_1 + \cdots + (c_n - c_n') v_n = 0$$

$\Rightarrow$ $c_i = c_i'$ since $\{v_1 \cdots v_n\}$ are lin indep't.

#3



#4 a) $Ax = b \iff LUx = b$ so solve

$Lc = b$ & $Ux = c$

b) $Lc = b \iff \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ $c_1 = 1$

$c_1 = 1$

$2c_1 + c_2 = 2 \iff c_2 = 2 - 2 = 0$

$-c_1 + c_2 + c_3 = -1 \iff c_3 = 1 - 1 = 0$

$Ux = c \iff \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$-x_3 = 0 \quad x_3 = 0$

$-x_2 + x_3 = 0 \quad x_2 = 0$

$2x_1 + x_2 + 2x_3 = 1 \quad x_1 = 1/2$

#5 $A = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & 3 & 0 & 1 \\ -2 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\ell_{21}=2} \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ -2 & 1 & 1 & 0 \end{bmatrix}$

$$\Big\downarrow \ \ell_{31} = -1$$

$$U = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -4 & 5 \end{bmatrix} \xleftarrow{\ell_{32}=2} \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = U$$

$$L = (\ell_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

(5)

#6 (a) Ask that $(b - A\hat{x}) \perp \text{Col}(A)$

$$\Leftrightarrow (b - A\hat{x})^T A = 0 \Leftrightarrow A^T(b - A\hat{x}) = 0$$

$$\Leftrightarrow A^T b = A^T A \hat{x} \Leftrightarrow \hat{x} = (A^T A)^{-1} A^T b$$

Thm: $(A^T A)^{-1}$ exists when $\text{rank}(A) = n$

(b) $\left\| b - \underbrace{\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \end{bmatrix}}_{A} \hat{x} \right\|^2$ minimized $\Rightarrow$

$\hat{x} = (C, D)$

$$(A^T A)\hat{x} = A^T b \qquad A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Need: $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$$(A^T A)^{-1} = \frac{1}{8}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \qquad \begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{\cancel{8}_4}\begin{bmatrix} \cancel{6}^{3} \\ -\cancel{2}_{-1} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$