Section 3.3

1.) $y = x^2 - x - 2 \xrightarrow{D} y' = 2x - 1 \xrightarrow{D} y'' = 2$

$\underline{+ \quad + \quad +}\ y''$; y is $\cup$ for all x-values

6.) $f(x) = \dfrac{x^2}{x^2+1} \xrightarrow{D} f'(x) = \dfrac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2}$

$= \dfrac{2x\left[(\cancel{x^2}+1) - \cancel{x^2}\right]}{(x^2+1)^2} = \dfrac{2x}{(x^2+1)^2} \xrightarrow{D}$

$f''(x) = \dfrac{(x^2+1)^2(2) - (2x)\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4}$

$= \dfrac{2(x^2+1)\cdot\left[(x^2+1) - 4x^2\right]}{(x^2+1)^4} = \dfrac{2(1-3x^2)}{(x^2+1)^3} = 0$

$\rightarrow x = \frac{1}{\sqrt{3}},\ x = \frac{-1}{\sqrt{3}}$: $\underline{- \quad 0 \quad + \quad 0 \quad -}\ f''$

$x = \frac{-1}{\sqrt{3}} \qquad x = 1/\sqrt{3}$

f is $\cup$ for $\frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$;
f is $\cap$ for $x < \frac{-1}{\sqrt{3}}$, $x > \frac{1}{\sqrt{3}}$.

7.) $y = -x^3 + 6x^2 - 9x - 1 \xrightarrow{D} y' = -3x^2 + 12x - 9 \xrightarrow{D}$

$y'' = -6x + 12 = 0 \rightarrow x = 2$: $\underline{+ \quad 0 \quad -}\ y''$

$x = 2$

y is $\cup$ for $x < 2$;
y is $\cap$ for $x > 2$.

15.) $f(x) = \sqrt{x^2+1}$, Domain : all x-values ;

$\xrightarrow{D} f'(x) = \frac{1}{2}(x^2+1)^{-1/2}\cdot(2x) = \dfrac{x}{\sqrt{x^2+1}} = 0 \rightarrow x = 0$:

$\underline{- \quad 0 \quad +}\ f$

$\left.\begin{matrix} x=0 \\ y=1 \end{matrix}\right\}$ abs. min.

f is $\uparrow$ for $x > 0$;
f is $\downarrow$ for $x < 0$.

$$\xrightarrow{D} f''(x) = \frac{\sqrt{x^2+1} - x \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot (2x)}{(x^2+1)}$$

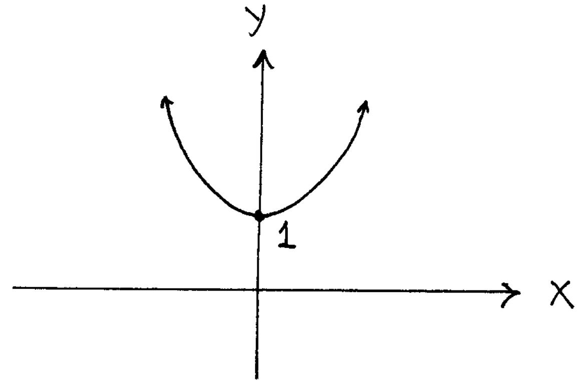
$$= \frac{\frac{\sqrt{x^2+1}}{1} - \frac{x^2}{\sqrt{x^2+1}}}{\frac{x^2+1}{1}} = \frac{(\cancel{x^2}+1)-\cancel{x^2}}{(x^2+1)^{1/2}} \cdot \frac{1}{x^2+1} = \frac{1}{(x^2+1)^{3/2}} :$$

$+ \quad + \quad +$ f''

f is ∪ for all x-values ;

$x=0: \ y=1$,

$y=0: \ (NO)$



27.) $f(x) = x^3 - 9x^2 + 24x - 18$, Domain : all x-values ;

$$\xrightarrow{D} f'(x) = 3x^2 - 18x + 24 = 3(x^2-6x+8)$$

$$= 3(x-4)(x-2) = 0 \longrightarrow x=4, \ x=2 :$$

$+ \ 0 \ - \ 0 \ +$ f'

$x=2$, $\underbrace{y=2}_{\text{rel. max.}}$; $x=4$, $\underbrace{y=-2}_{\text{rel. min.}}$

f is ↑ for $x<2, x>4$;

f is ↓ for $2<x<4$;

$$\xrightarrow{D} f''(x) = 3(2x-6) = 0 \rightarrow x=3 :$$

$- \ 0 \ +$ f''

$\left.\begin{matrix} x=3 \\ y=0 \end{matrix}\right\}$ infl. pt.

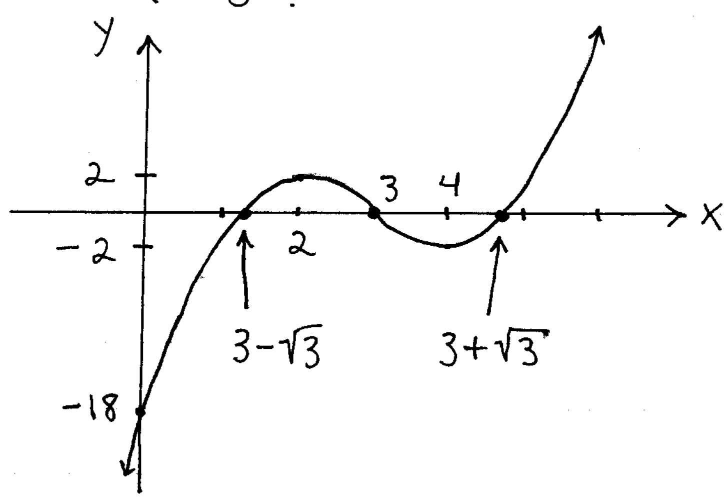
f is ∪ for $x>3$;

f is ∩ for $x<3$;

$x=0: \ y=-18$

$y=0: \ (x-3)(x^2-6x+6)=0$

$\rightarrow x=3, \ x = 3 \pm \sqrt{3}$



29.) $f(x) = (x-1)^3(x-5)$, Domain: all x-values;

$\xrightarrow{D} f'(x) = (x-1)^3 \cdot (1) + 3(x-1)^2 \cdot (x-5)$

$= (x-1)^2[(x-1)+3(x-5)] = (x-1)^2 \cdot [4x-16] = 0 \rightarrow$

$x=1$, $x=4$:

$-\ 0\ -\ 0\ +$ f'

$x=1$ $\quad x=4$, $y=-27$ } abs. min.

f is ↑ for $x>4$;

f is ↓ for $x<1$, $1<x<4$;

$\xrightarrow{D} f''(x) = (x-1)^2(4) + 2(x-1)(4x-16)$

$= 4(x-1)[(x-1)+2(x-4)] = 4(x-1)(3x-9) = 0 \rightarrow$

$x=1$, $x=3$:

$+\ 0\ -\ 0\ +$ f''

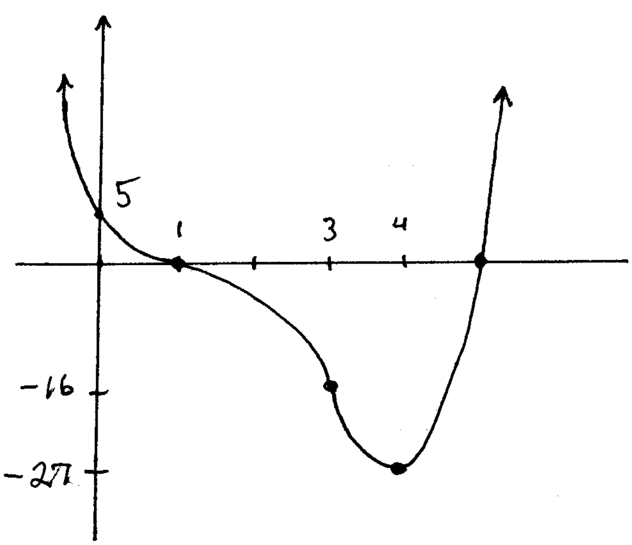
$\underbrace{x=1,\ y=0 \quad x=3,\ y=-16}_{\text{infl. pts.}}$

f is ∪ for $x<1$, $x>3$;

f is ∩ for $1<x<3$;

$x=0$: $y=5$

$y=0$: $x=1$, $x=5$



35.) $f(x) = x^3 - 12x$, Domain: all x-values;

$\xrightarrow{D} f'(x) = 3x^2 - 12 = 3(x-2)(x+2) = 0 \rightarrow x=2$, $x=-2$:

$+\ 0\ -\ 0\ +$ f'

rel. max. { $x=-2$, $y=16$ $\quad x=2$, $y=-16$ } rel. min.

f is ↑ for $x<-2$, $x>2$;

f is ↓ for $-2<x<2$;

$\xrightarrow{D} f''(x) = 6x = 0 \rightarrow x=0$:

$-\ 0\ +$ f''

infl. pt. { $x=0$, $y=0$

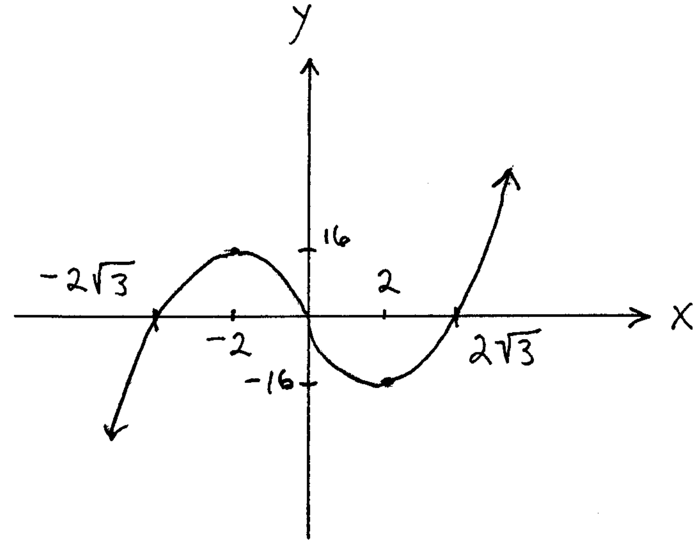
f is ∪ for $x>0$;

f is ∩ for $x<0$;

$x=0: \ y=0$,

$y=0: \ x(x^2-12)=0$

$\rightarrow x=0, \ x=\pm 2\sqrt{3}$



39.) $f(x) = \frac{1}{4}x^4 - 2x^2$, Domain: all x-values;

$\xrightarrow{D} f'(x) = x^3 - 4x = x(x-2)(x+2) = 0 \rightarrow x=2, \ x=-2, \ x=0:$

$-$	0	$+$	0	$-$	0	$+$	f'
	$x=-2$		$x=0$		$x=2$		
	$\underbrace{y=-4}$		$\underbrace{y=0}$		$\underbrace{y=-4}$		
	abs. min.		rel. max.		abs. min.		

f is $\uparrow$ for $-2<x<0, \ x>2$;

f is $\downarrow$ for $x<-2, \ 0<x<2$;

$\xrightarrow{D} f''(x) = 3x^2 - 4 = 3\left(x^2 - \frac{4}{3}\right) = 0 \rightarrow x = \pm\frac{2}{\sqrt{3}}$:

$+$	0	$-$	0	$+$	f''
	$x=\frac{-2}{\sqrt{3}}$		$x=\frac{2}{\sqrt{3}}$		
	$y=\frac{-20}{9}$		$y=\frac{-20}{9}$		

$\underbrace{\hspace{4em}}_{\text{infl. pts.}}$

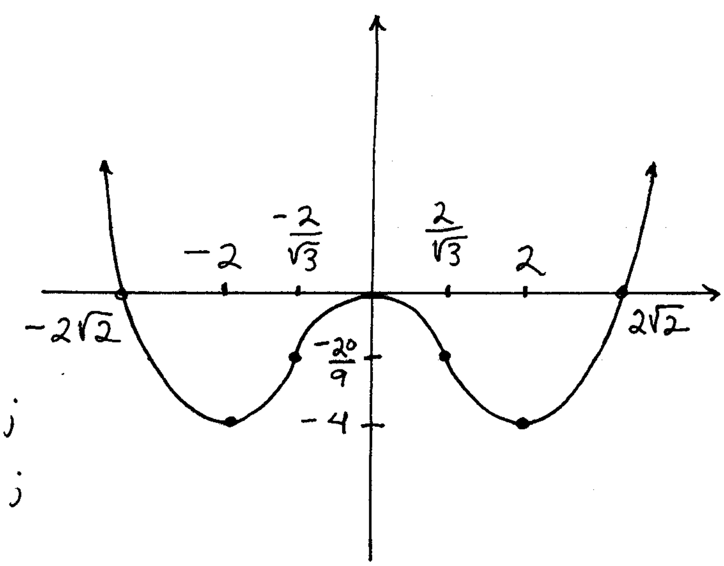
f is $\cup$ for $x<\frac{-2}{\sqrt{3}}, \ x>\frac{2}{\sqrt{3}}$;

f is $\cap$ for $\frac{-2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$;

$x=0: \ y=0$,

$y=0: \ \frac{1}{4}x^2(x^2-8) = 0$

$\rightarrow x=0, \ x=\pm 2\sqrt{2}$



44.) $g(x) = x\sqrt{9-x}$, Domain: $x \le 9$;

$g'(x) = x \cdot \frac{1}{2}(9-x)^{-1/2}(-1) + \sqrt{9-x} = \frac{-x}{2\sqrt{9-x}} + \frac{\sqrt{9-x}}{1}$

$= \frac{-x + 2(9-x)}{2\sqrt{9-x}} = \frac{18-3x}{2\sqrt{9-x}} = 0 \longrightarrow x = 6$:

+ 0 − NO

$x=6$ $x=9$

$y=6\sqrt{3}$ $y=0$

abs. max. rel. min.

f is $\uparrow$ for $x < 6$;

f is $\downarrow$ for $6 < x < 9$;

$\xrightarrow{D} g''(x) = \frac{2\sqrt{9-x}\cdot(-3) - 3(6-x)\cdot\cancel{2}\cdot\frac{1}{\cancel{2}}(9-x)^{-1/2}\cdot(-1)}{4(9-x)}$

$= \frac{-3\left(\frac{2\sqrt{9-x}}{1} - \frac{6-x}{\sqrt{9-x}}\right)}{\frac{4(9-x)}{1}} = -3 \cdot \frac{2(9-x) - 6 + x}{\sqrt{9-x}} \cdot \frac{1}{4(9-x)}$

$= \frac{-3}{4} \cdot \frac{12-x}{(9-x)^{3/2}} = 0 \longrightarrow x = 12$ (NO!)

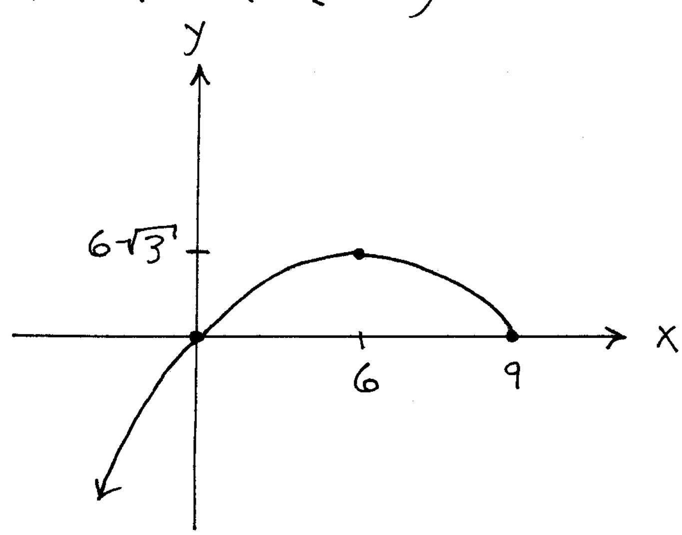
− − − NO f''

$x=9$

f is $\cap$ for $x < 9$;

$x=0$: $y=0$,

$y=0$: $x=0$, $x=9$



46.) $f(x) = \dfrac{2}{x^2-1}$, Domain : all x but $x=1, x=-1$;

$\xrightarrow{D} f'(x) = \dfrac{(x^2-1)(0) - 2(2x)}{(x^2-1)^2} = \dfrac{-4x}{(x^2-1)^2} = 0 \to x = 0$:

NO | NO

+ | + 0 − | − f'

$x=-1$ $x=0$ $x=1$

$\underbrace{y=-2}_{\text{rel. max.}}$

f is ↑ for $x<-1$, $-1<x<0$;

f is ↓ for $0<x<1$, $x>1$;

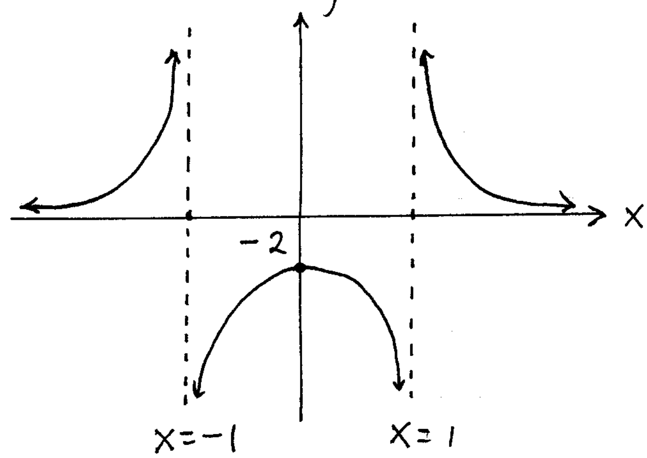
$\xrightarrow{D} f''(x) = \dfrac{(x^2-1)^2(-4) - (-4x)\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4}$

$= \dfrac{-4(x^2-1)\cdot[(x^2-1) - 4x^2]}{(x^2-1)^4} = \dfrac{-4(-1-x^2)}{(x^2-1)^3} = \dfrac{4(1+x^2)}{(x^2-1)^3}$:

NO | NO

+ | − | + f''

$x=-1$ $x=1$



f is ∪ for $x<-1$, $x>1$;

f is ∩ for $-1<x<1$;

$x=0$: $y=-2$,

$y=0$: (NO) ;

$\displaystyle\lim_{x\to\pm\infty} \frac{2}{x^2-1} = \text{"}\frac{2}{\infty}\text{"} = 0 \to$ H.A. is $y=0$;

$\left.\begin{aligned} &\lim_{x\to1^+} \frac{2}{x^2-1} = \text{"}\frac{2}{0^+}\text{"} = +\infty \\ &\lim_{x\to1^-} \frac{2}{x^2-1} = \text{"}\frac{2}{0^-}\text{"} = -\infty \end{aligned}\right\}$ V.A. is $x=1$;

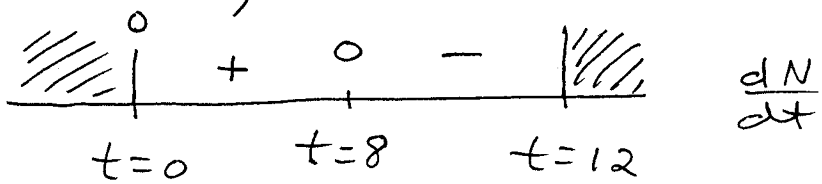
$\left.\begin{aligned} &\lim_{x\to-1^+} \frac{2}{x^2-1} = \text{"}\frac{2}{0^-}\text{"} = -\infty \\ &\lim_{x\to-1^-} \frac{2}{x^2-1} = \text{"}\frac{2}{0^+}\text{"} = +\infty \end{aligned}\right\}$ V.A. is $x=-1$.

65.) $N = -t^3 + 12t^2$, $0 \le t \le 12$

a.)

$\xrightarrow{D} \frac{dN}{dt} = -3t^2 + 24t = 3t(8-t) = 0 \rightarrow$

$t = 0$ wks., $t = 8$ wks. :

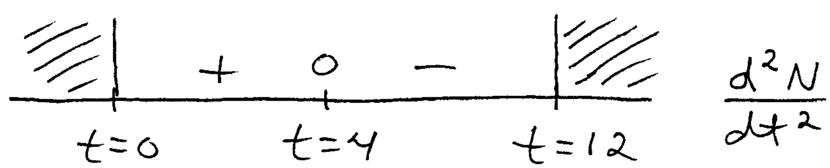


$N = 256$ (100's)

So maximum # of people to be infected is 25,600 at week 8 ;

b.) When is rate $\frac{dN}{dt}$ maximum?

$\xrightarrow{D} D\left(\frac{dN}{dt}\right) = \frac{d^2N}{dt^2} = -6t + 24 = 0 \rightarrow t = 4$ wks.:



$\frac{dN}{dt} = 48$ (100's) per week

So maximum rate is 4800 people/wk. at week 4 .