Section 3.7

1.) $Y = -x^2 - 2x + 3$ domain : all x-values

$Y' = -2x - 2 = -2(x+1) = 0$

$+ \quad 0 \quad -$ Y'

$x = -1$
$Y = 4$ } abs. max.

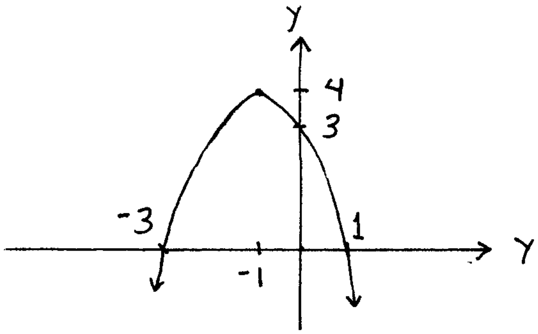
$Y'' = -2$

$- \quad - \quad -$ Y''

Y is ↑ for $x < -1$,
Y is ↓ for $x > -1$,
Y is ∩ for all x-values

$x = 0 : \ Y = 3$

$Y = 0 : \ (-x + 1)(x + 3) = 0$
$\downarrow \quad \downarrow$
$x = 1 \quad x = -3$



2.) $Y = 2x^2 - 4x + 1$ domain : all x-values

$Y' = 4x - 4 = 4(x-1) = 0$

$- \quad 0 \quad +$ Y'

$x = 1$
$Y = -1$ } abs. min.

$Y'' = 4$

$+ \quad + \quad +$ Y''

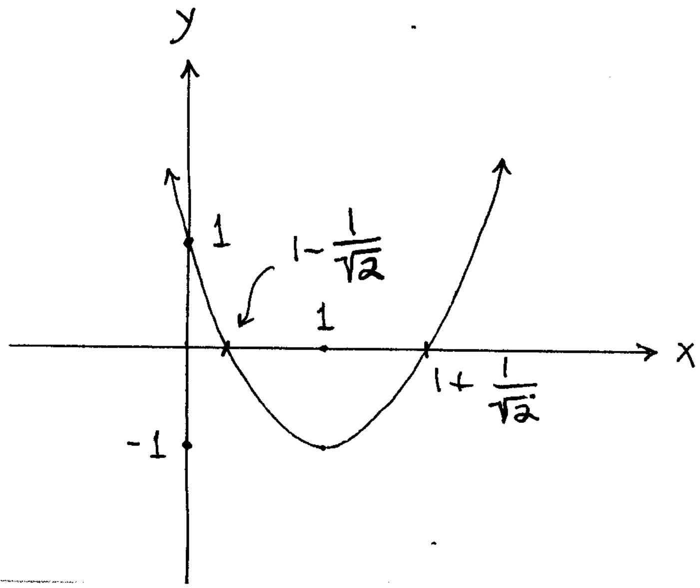
Y is ↑ for $x > 1$,
Y is ↓ for $x < 1$,
Y is ∪ for all x-values

$x = 0 : \ Y = 1$

$Y = 0 : \ 2x^2 - 4x + 1 = 0 \rightarrow$

$$x = \frac{4 \pm \sqrt{8}}{4} = 1 \pm \frac{1}{\sqrt{2}}$$

Y is ↑ for $x > 1$,
Y is ↓ for $x < 1$,
Y is ∪ for all x-values



9.) $Y = 3X^4 + 4X^3$ domain : all x-values

$Y' = 12X^3 + 12X^2$

$= 12X^2(X+1) = 0$

− 0 + 0 + Y'

abs. min. $\left\{ \begin{array}{l} X=-1 \\ Y=-1 \end{array} \right.$ $\left. \begin{array}{l} X=0 \\ Y=0 \end{array} \right\}$?

$Y'' = 36X^2 + 24X$

$= 12X(3X+2) = 0$

+ 0 − 0 + Y''

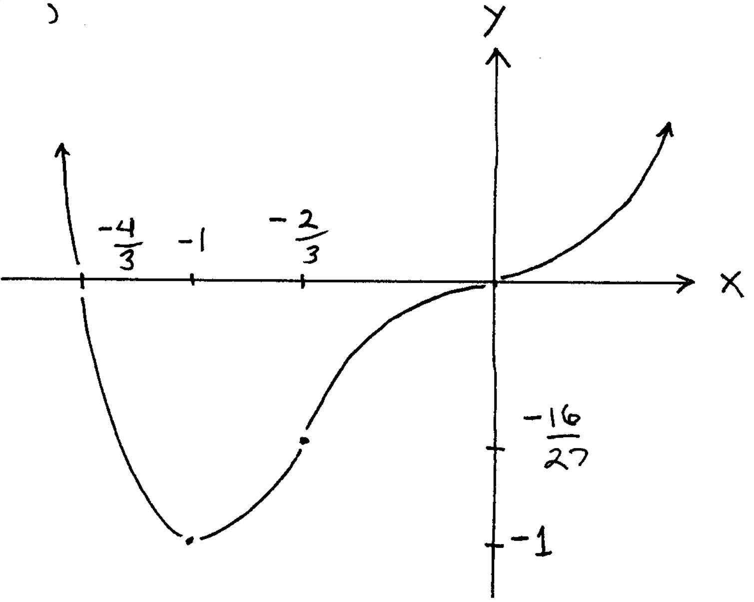
inf. pt. $\left\{ \begin{array}{l} X=\frac{-2}{3} \\ Y=\frac{-16}{27} \end{array} \right.$ $\left. \begin{array}{l} X=0 \\ Y=0 \end{array} \right\}$ inf. pt.

Y is ↑ for $-1 < X < 0$, $X > 0$,

Y is ↓ for $X < -1$,

Y is ∪ for $X < {-2/3}$, $X > 0$,

Y is ∩ for $-2/3 < X < 0$



$X = 0 : Y = 0$

$Y = 0 : 3X^4 + 4X^3 = 0$

$X^3(3X+4) = 0$

↓ ↓

$X = 0$ $X = \frac{-4}{3}$

17.) $Y = X^5 - 5X$ domain : all x-values

$Y' = 5X^4 - 5 = 5(X^2-1)(X^2+1)$

$= 5(X-1)(X+1)(X^2+1) = 0$

+ 0 − 0 + Y'

rel. max. $\left\{ \begin{array}{l} X=-1 \\ Y=4 \end{array} \right.$ $\left. \begin{array}{l} X=1 \\ Y=-4 \end{array} \right\}$ rel. min.

$Y'' = 20X^3 = 0$

− 0 + Y''

$\left. \begin{array}{l} X=0 \\ Y=0 \end{array} \right\}$ inf. pt.

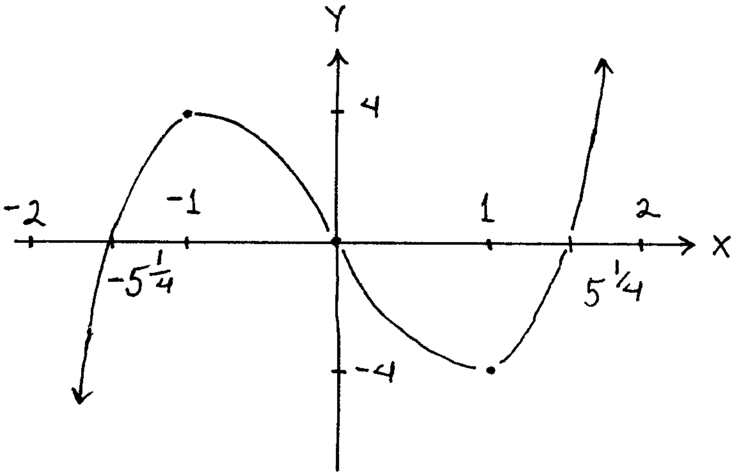
Y is ↑ for $X<-1$, $X>1$,
Y is ↓ for $-1<X<1$,
Y is ∪ for $X>0$,
Y is ∩ for $X<0$.

$X=0$: $Y=0$

$Y=0$: $X^5-5X=0$

$$\rightarrow X\left(X^2-\sqrt{5}\right)\left(X^2+\sqrt{5}\right)=0$$

↓ $X=0$ ↘ $X=\pm 5^{1/4}$



22.) $Y=\dfrac{X}{X^2+1}$ domain: all x-values

$$Y'=\frac{(X^2+1)(1)-X(2X)}{(X^2+1)^2}=\frac{1-X^2}{(X^2+1)^2}=0$$

$-\ \ 0\ \ +\ \ 0\ \ -$ Y'

abs. min. $\begin{cases}X=-1\\Y=-\frac{1}{2}\end{cases}$ $\begin{matrix}X=1\\Y=\frac{1}{2}\end{matrix}\Big\}$ abs. max.

$$Y''=\frac{(X^2+1)^2(-2X)-(1-X^2)\cdot 2(X^2+1)\cdot 2X}{(X^2+1)^4}$$

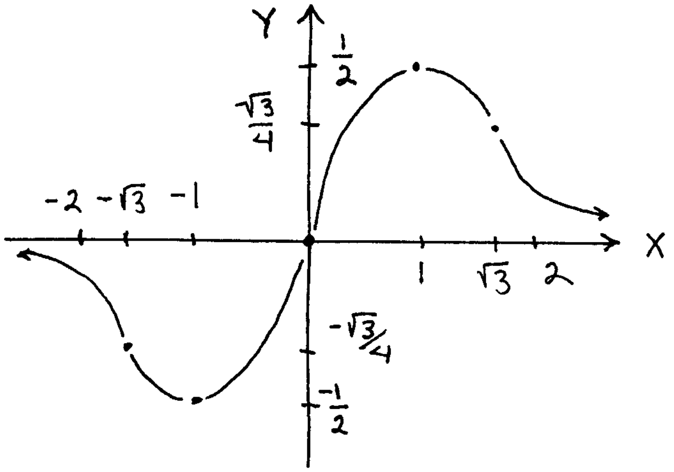
$$=\frac{-2X(X^2+1)\left[(X^2+1)+2(1-X^2)\right]}{(X^2+1)^4}$$

$$=\frac{-2X(3-X^2)}{(X^2+1)^3}=0$$

all are inf. pts.
↙ ↓ ↘

$-\ \ 0\ \ +\ \ 0\ \ -\ \ 0\ \ +$ Y''

$X=-\sqrt{3}$, $Y=-\sqrt{3}/4$ $X=0$, $Y=0$ $X=\sqrt{3}$, $Y=\sqrt{3}/4$



Y is ↑ for $-1<X<1$,
Y is ↓ for $X<-1$, $X>1$,
Y is ∪ for $-\sqrt{3}<X<0$, $X>\sqrt{3}$,
Y is ∩ for $X<-\sqrt{3}$, $0<X<\sqrt{3}$.

$X=0$: $Y=0$

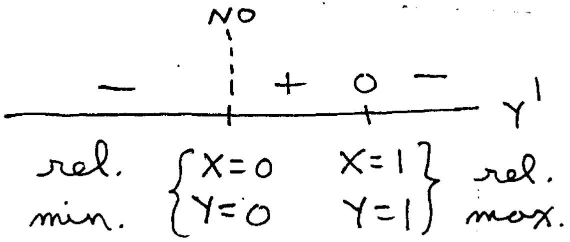
$Y=0$: $X=0$

$$\lim_{X\to\pm\infty}\frac{X}{X^2+1}=\lim_{X\to\pm\infty}\frac{\frac{1}{X}}{1+\frac{1}{X^2}}=0$$: H.A. is $y=0$.

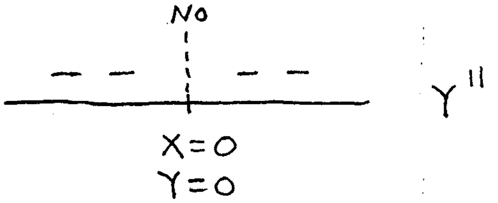
23.) $Y = 3X^{2/3} - 2X$ domain : X- values

$Y' = 3\cdot\frac{2}{3}X^{-1/3} - 2 = 2X^{-1/3} - 2$

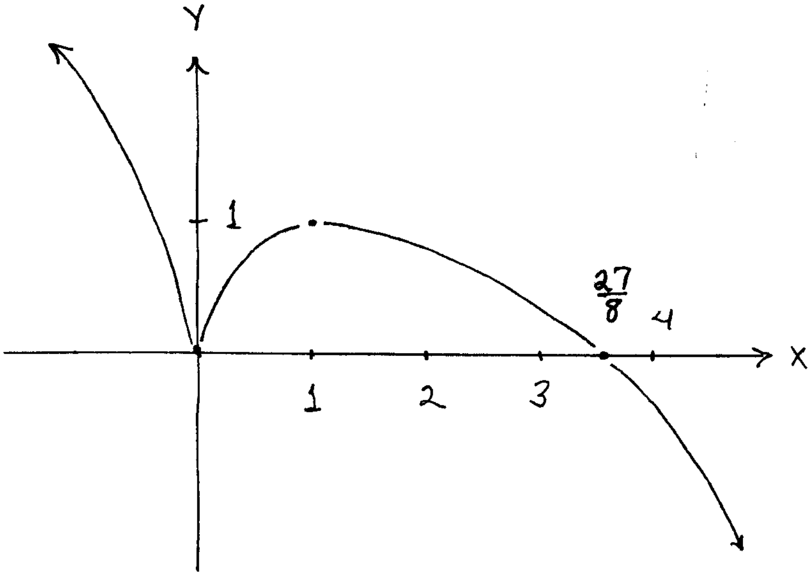
$= 2\left(\frac{1}{X^{1/3}} - 1\right) = 2\left(\frac{1 - X^{1/3}}{X^{1/3}}\right) = 0$



$Y'' = 2\cdot\frac{-1}{3}X^{-4/3} = \frac{-2}{3X^{4/3}}$



Y is ↑ for $0 < x < 1$,
Y is ↓ for $x < 0$, $x > 1$,
Y is ∩ for $x < 0$, $x > 0$



$X = 0 : Y = 0$

$Y = 0 : \ 3X^{2/3} - 2X = 0$

$\rightarrow X^{2/3}\left(3 - 2X^{1/3}\right) = 0$

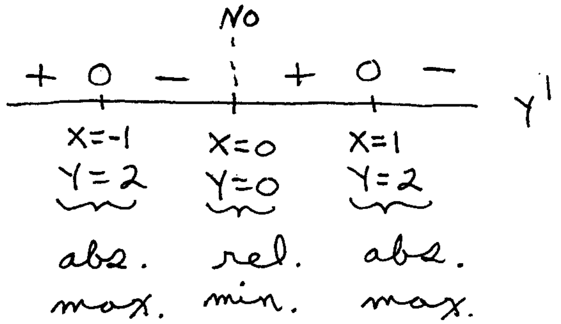
$\downarrow \qquad \downarrow$

$X = 0 \qquad X = \frac{27}{8}$

24.) $Y = 3X^{2/3} - X^2$ domain : all X- values

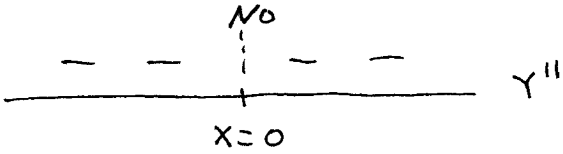
$Y' = 3\cdot\frac{2}{3}X^{-1/3} - 2X = 2X^{-1/3} - 2X$

$= 2\left(\frac{1}{X^{1/3}} - \frac{X}{1}\right) = 2\left(\frac{1 - X^{4/3}}{X^{1/3}}\right) = 0$



$Y'' = 2\cdot\frac{-1}{3}X^{-\frac{4}{3}} - 2 = \frac{-2}{3X^{4/3}} - \frac{2}{1}$

$= -2\left(\frac{1 + 3X^{4/3}}{3X^{4/3}}\right) = 0$



Y is ↑ for $x<-1$, $0<x<1$,

Y is ↓ for $-1<x<0$, $x>1$,

Y is ∩ for $x<0$, $x>0$

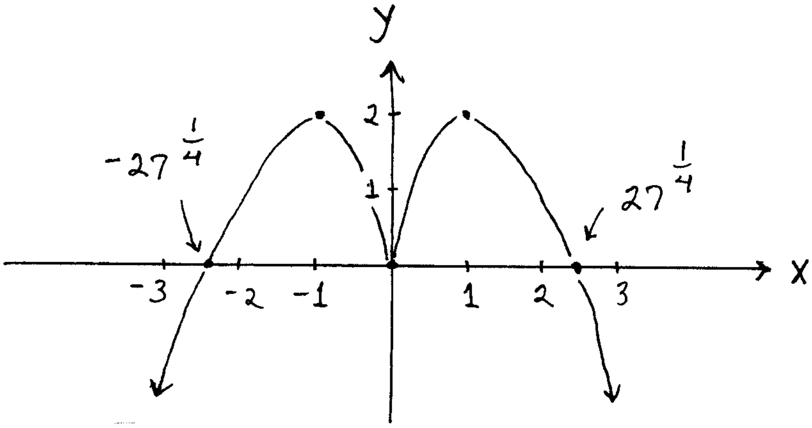
$x=0$: $Y=0$

$Y=0$: $3x^{2/3}-x^2=0$

$$x^{2/3}\left(3-x^{4/3}\right)=0$$

$$\downarrow \qquad\qquad \downarrow$$

$$x=0 \qquad x=\pm 27^{\frac{1}{4}}$$



34.) $Y=\dfrac{x^2+1}{x^2-2}$ domain : all $x \neq \pm\sqrt{2}$

$$Y'=\frac{(x^2-2)(2x)-(x^2+1)(2x)}{(x^2-2)^2}=\frac{-6x}{(x^2-2)^2}=0$$

+ ¦ + 0 − ¦ − Y'

$x=-\sqrt{2}$ $x=0$ $x=\sqrt{2}$

$Y=\frac{-1}{2}$ rel. max.

$$Y''=\frac{(x^2-2)^2(-6)-(-6x)\cdot 2(x^2-2)\cdot 2x}{(x^2-2)^4}$$

$$=\frac{-6(x^2-2)\cdot\left[(x^2-2)-4x^2\right]}{(x^2-2)^4}=\frac{6(2+3x^2)}{(x^2-2)^3}=0$$

+ ¦ − ¦ + Y''

$x=-\sqrt{2}$ $x=\sqrt{2}$

Y is ↑ for $x<-\sqrt{2}$, $-\sqrt{2}<x<0$,

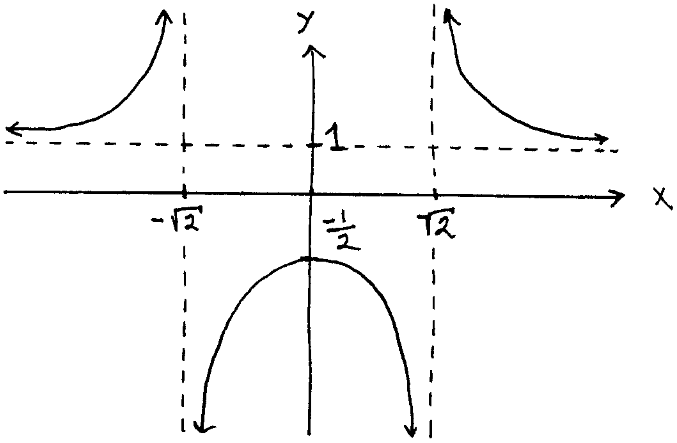
Y is ↓ for $0<x<\sqrt{2}$, $x>\sqrt{2}$,

Y is ∪ for $x<-\sqrt{2}$, $x>\sqrt{2}$,

Y is ∩ for $-\sqrt{2}<x<\sqrt{2}$

$x=0$: $Y=-1/2$

$Y=0$: none



$$\lim_{x\to+\sqrt{2}^+}\frac{x^2+1}{x^2-2}=+\infty,\quad \lim_{x\to+\sqrt{2}^-}\frac{x^2+1}{x^2-2}=-\infty$$

$$\lim_{x\to-\sqrt{2}^+}\frac{x^2+1}{x^2-2}=-\infty,\quad \lim_{x\to-\sqrt{2}^-}\frac{x^2+1}{x^2-2}=+\infty$$

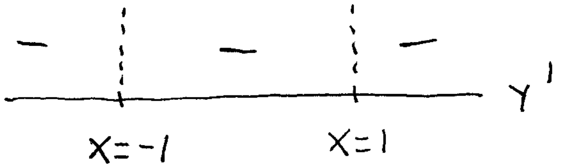
} vertical asymptotes: $x=\pm\sqrt{2}$

$$\lim_{x\to\pm\infty}\frac{x^2+1}{x^2-2}=\lim_{x\to\pm\infty}\frac{1+\frac{1}{x^2}}{1-\frac{2}{x^2}}=1$$

: horizontal asymptote: $Y=1$

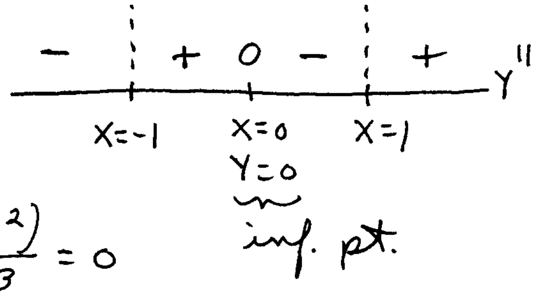
35. $Y = \frac{2x}{x^2-1}$ domain : all $x \neq \pm 1$

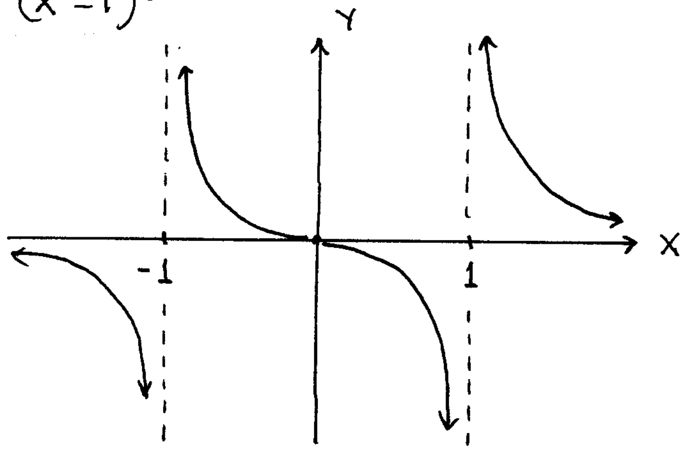
$$Y' = \frac{(x^2-1)(2)-(2x)(2x)}{(x^2-1)^2} = \frac{-2(1+x^2)}{(x^2-1)^2} = 0$$



$$Y'' = \frac{(x^2-1)^2\cdot(-4x)+2(1+x^2)\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4}$$

$$= \frac{-4x(x^2-1)\cdot[(x^2-1)-2(1+x^2)]}{(x^2-1)^4} = \frac{4x(3+x^2)}{(x^2-1)^3} = 0$$





Y is ↓ for $x<-1$, $-1<x<1$, $x>1$,

Y is ∪ for $-1<x<0$, $x>1$,

Y is ∩ for $x<-1$, $0<x<1$

$X=0: Y=0$ and $Y=0: X=0$

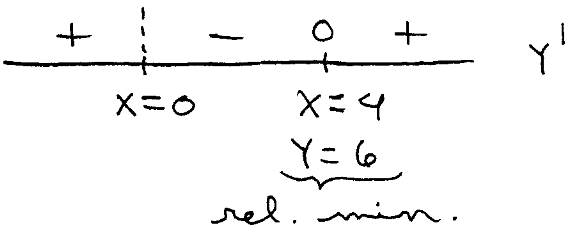
$$\lim_{x\to 1^+} \frac{2x}{x^2-1} = +\infty, \quad \lim_{x\to 1^-} \frac{2x}{x^2-1} = -\infty$$

$$\lim_{x\to -1^+} \frac{2x}{x^2-1} = +\infty, \quad \lim_{x\to -1^-} \frac{2x}{x^2-1} = -\infty$$

} vertical asymptotes

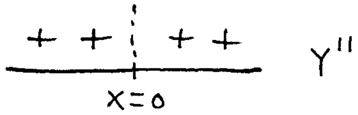
$$\lim_{x\to\pm\infty} \frac{2x}{x^2-1} = \lim_{x\to\pm\infty} \frac{\frac{2}{x}}{1-\frac{1}{x^2}} = 0$$: horizontal asymptote

40.) $Y = X + \frac{32}{X^2}$ domain : all $x \neq 0$



$Y' = 1 - \frac{64}{X^3} = \frac{X^3 - 64}{X^3} = 0$

$Y'' = \frac{192}{X^4}$



Y is ↑ for $X<0, X>4$

Y is ↓ for $0<X<4$

Y is ∪ for $X<0, X>0$

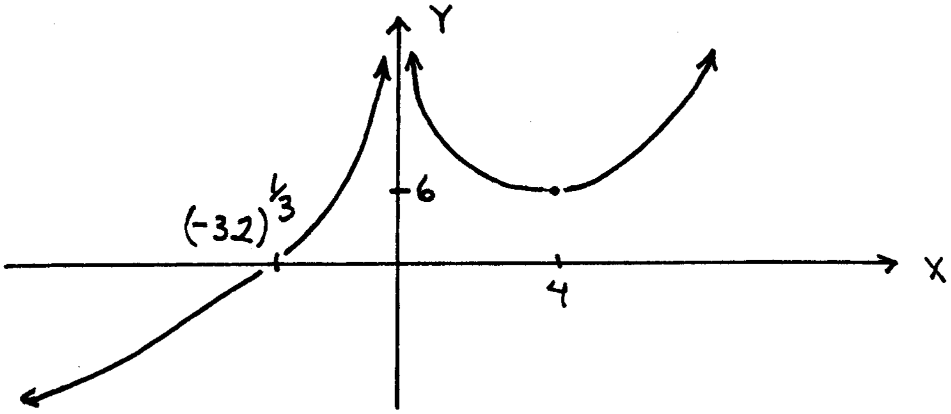
$Y=0 \rightarrow 0 = X + \frac{32}{X^2}$

$\rightarrow X = (-32)^{1/3} \approx -3.17$

$\lim_{X \to 0^{\pm}} \left(X + \frac{32}{X^2}\right) = \text{"}\frac{1}{0^+}\text{"} = +\infty$ so vertical asymptote at $X=0$;

$\lim_{X \to +\infty} \left(X + \frac{32}{X^2}\right) = +\infty$;

$\lim_{X \to -\infty} \left(X + \frac{32}{X^2}\right) = -\infty$.



Section 3.2

22.) $f(x) = x^2 + 2x - 4$ on $[-1, 1]$

$f'(x) = 2x + 2$

$= 2(x+1) = 0$

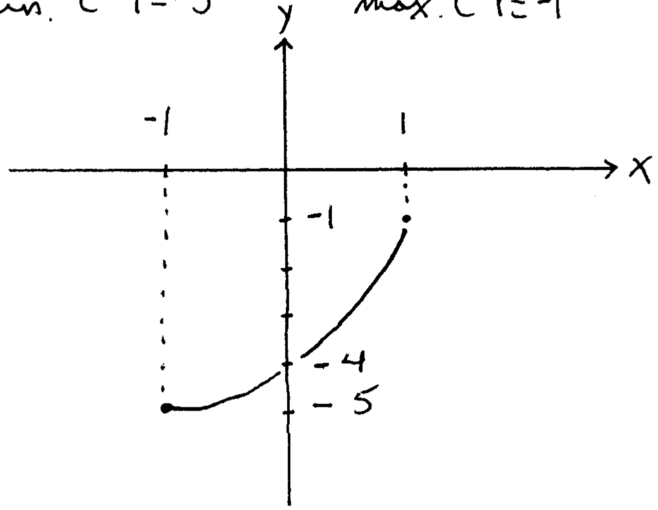
$0 \quad + \quad + \quad f'$

abs. min. $\begin{cases} x = -1 \\ Y = -5 \end{cases}$	abs. max. $\begin{cases} X = 1 \\ Y = -1 \end{cases}$

$f''(x) = 2$

$+ \quad + \quad f''$

$X = -1 \qquad X = 1$



f is ↑ for $-1 < x < 1$,

f is ∪ for $-1 < x < 1$.

$x = 0$: $Y = -4$

$Y = 0$: not in $[-1, 1]$

23.) $f(x) = x^3 - 3x^2$ on $[-1, 3]$

$f'(x) = 3x^2 - 6x$

$= 3x(x-2) = 0$

$+ \quad 0 \quad - \quad 0 \quad + \quad f'$

X=-1	X=0	X=2	X=3
Y=-4	Y=0	Y=-4	Y=0
abs. min.	abs. max	abs. min.	abs. max.

$f''(x) = 6x - 6$

$= 6(x-1) = 0$

$- \quad 0 \quad + \quad f''$

$X = -1 \qquad X = 1 \qquad X = 3$

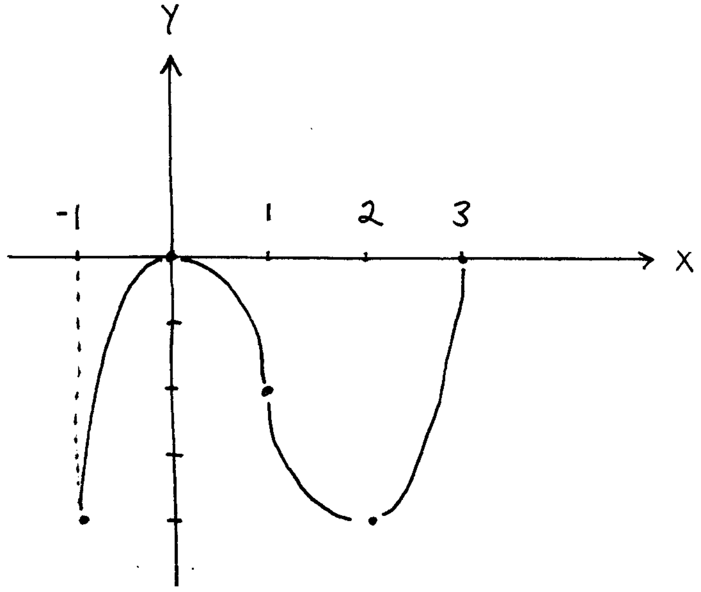
inf. pt. $\begin{cases} X = 1 \\ Y = -2 \end{cases}$

f is ↑ for $-1 < X < 0$, $2 < X < 3$,

f is ↓ for $0 < X < 2$,

f is ∪ for $1 < X < 3$,

f is ∩ for $-1 < X < 1$.



$X = 0$: $Y = 0$

$Y = 0$: $X^3 - 3X^2 = 0 \rightarrow$

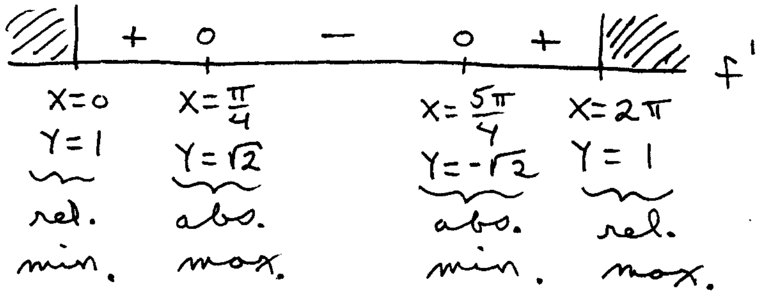
$X^2(X-3) = 0 \rightarrow \underline{X = 0}, X = 3$

I.) $f(x) = \sin x + \cos x$ on $[0, 2\pi]$

$f'(x) = \cos x - \sin x = 0$

$\rightarrow \cos x = \sin x$

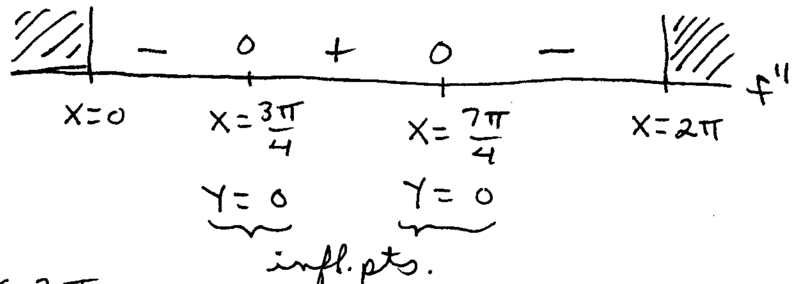
$\rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$



$f''(x) = -\sin x - \cos x = 0$

$\rightarrow -\sin x = \cos x$

$\rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$



f is ↑ for $0 < x < \frac{\pi}{4}$, $\frac{5\pi}{4} < x < 2\pi$,

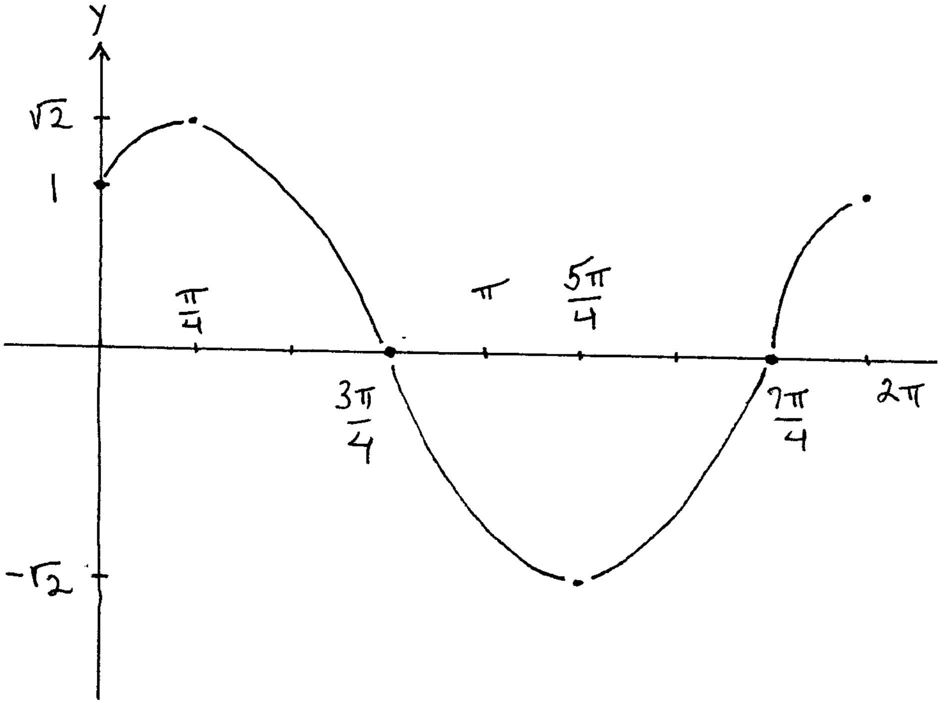
f is ↓ for $\frac{\pi}{4} < x < \frac{5\pi}{4}$,

f is ∪ for $\frac{3\pi}{4} < x < \frac{7\pi}{4}$,

f is ∩ for $0 < x < \frac{3\pi}{4}$, $\frac{7\pi}{4} < x < 2\pi$

$x = 0$: $Y = 1$

$Y = 0$: $\sin x + \cos x = 0 \rightarrow \sin x = {}^{-}\cos x \rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$

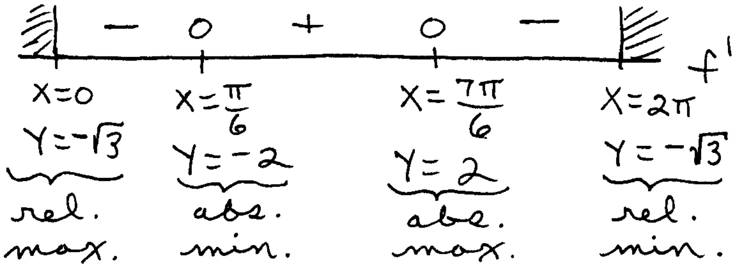


II.) $f(x) = -\sqrt{3}\cos x - \sin x$ on $[0, 2\pi]$

$f'(x) = \sqrt{3}\sin x - \cos x = 0$

$\rightarrow \frac{\cos x}{\sin x} = \sqrt{3}$

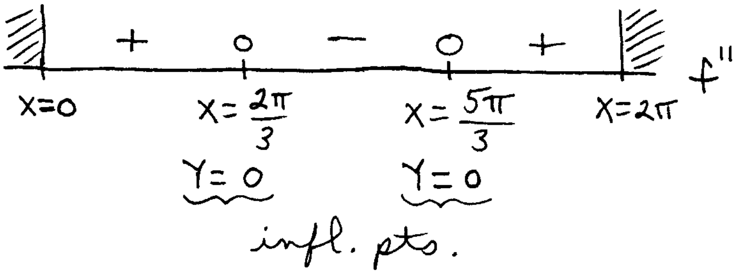
$\rightarrow x = \frac{\pi}{6}, \frac{7\pi}{6}$



$f''(x) = \sqrt{3}\cos x + \sin x = 0$

$\rightarrow \frac{\sin x}{\cos x} = -\sqrt{3}$

$\rightarrow x = \frac{2\pi}{3}, \frac{5\pi}{3}$



f is ↑ for $\frac{\pi}{6} < x < \frac{7\pi}{6}$,

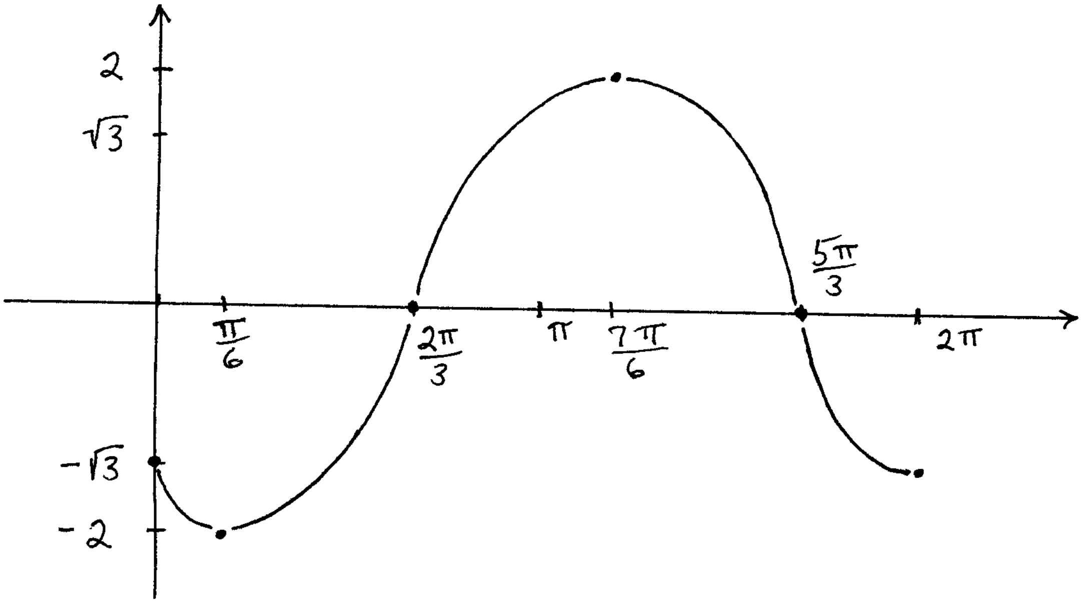
f is ↓ for $0 < x < \frac{\pi}{6}$, $\frac{7\pi}{6} < x < 2\pi$,

f is ∪ for $0 < x < \frac{2\pi}{3}$, $\frac{5\pi}{3} < x < 2\pi$,

f is ∩ for $\frac{2\pi}{3} < x < \frac{5\pi}{3}$.

$x = 0 \rightarrow y = -\sqrt{3}$

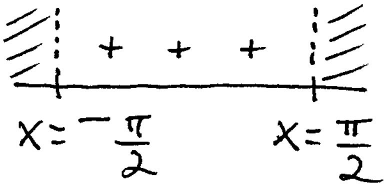
$y = 0 \rightarrow x = \frac{2\pi}{3}, \frac{5\pi}{3}$



Supplemental Trig

ST2.) $Y = \tan X$ on $-\frac{\pi}{2} < X < \frac{\pi}{2}$ $\xrightarrow{D}$

$Y' = \sec^2 X > 0$ for $-\frac{\pi}{2} < X < \frac{\pi}{2}$, so

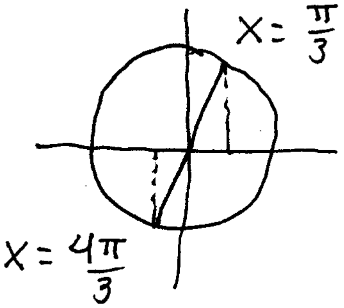


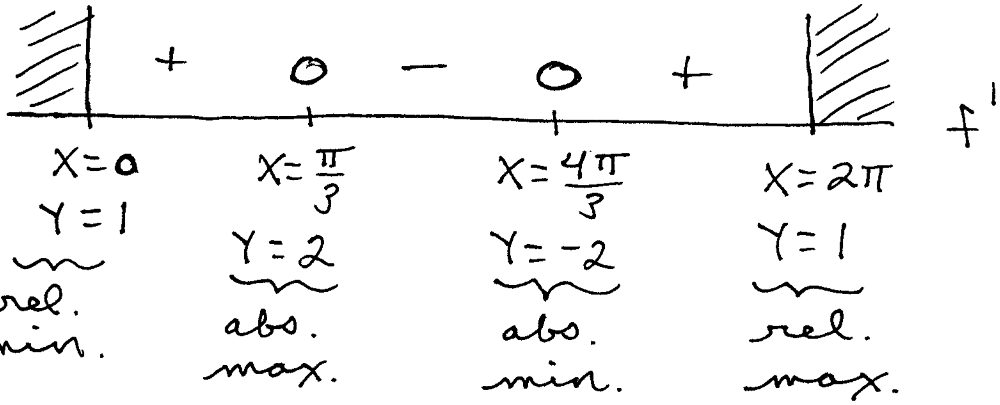
Y is ↑ on $-\frac{\pi}{2} < X < \frac{\pi}{2}$.

ST3.) a.) $f(x) = \sqrt{3}\sin X + \cos X$ on $[0, 2\pi]$

$\rightarrow f'(x) = \sqrt{3}\cos X - \sin X = 0 \rightarrow$

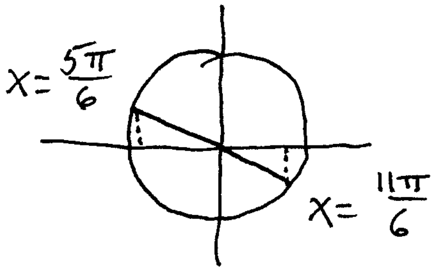
$\sqrt{3}\cos X = \sin X \rightarrow \sqrt{3} = \frac{\sin X}{\cos X} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$

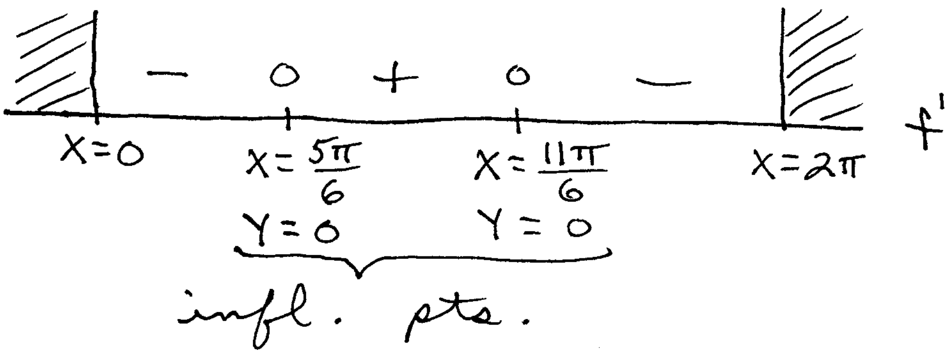




$f''(x) = -\sqrt{3}\sin X - \cos X = 0 \rightarrow$

$-\sqrt{3} = \frac{\cos X}{\sin X} = \frac{-\sqrt{3}/2}{1/2} = \frac{\sqrt{3}/2}{-1/2}$



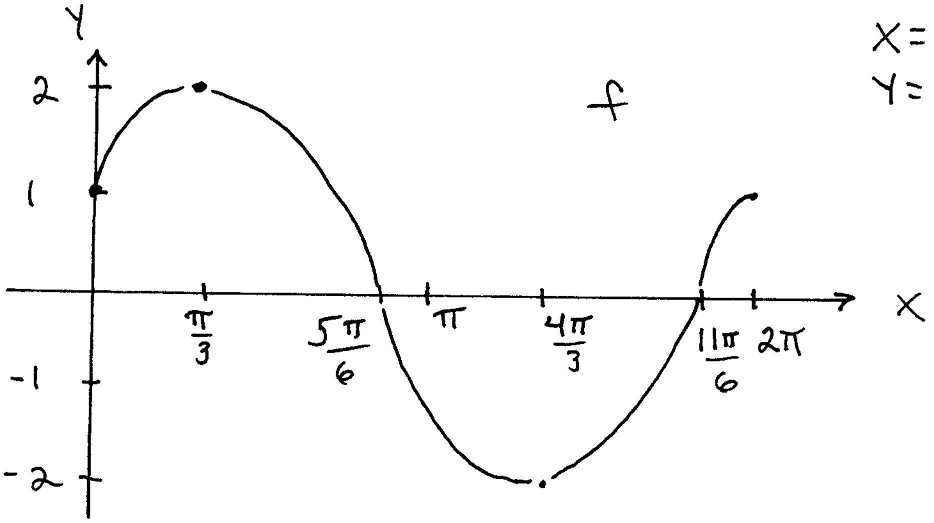


f is ↑ for $0 < x < \frac{\pi}{3}$, $\frac{4\pi}{3} < x < 2\pi$,

f is ↓ for $\frac{\pi}{3} < x < \frac{4\pi}{3}$,

f is ∪ for $\frac{5\pi}{6} < x < \frac{11\pi}{6}$,

f is ∩ for $0 < x < \frac{5\pi}{6}$, $\frac{11\pi}{6} < x < 2\pi$



$X = 0 : Y = 1$

$Y = 0 : X = \frac{5\pi}{6}, \frac{11\pi}{6}$

5T3.) b.) $f(x) = \sin^2 x$ on $[0, \pi]$

$\rightarrow f'(x) = 2 \sin x \cos x = 0 \rightarrow$

$\sin x = 0 \longrightarrow x = 0, \pi$ or $\cos x = 0 \rightarrow x = \frac{\pi}{2}$

$f''(x) = 2 \sin x \cdot (-\sin x)$

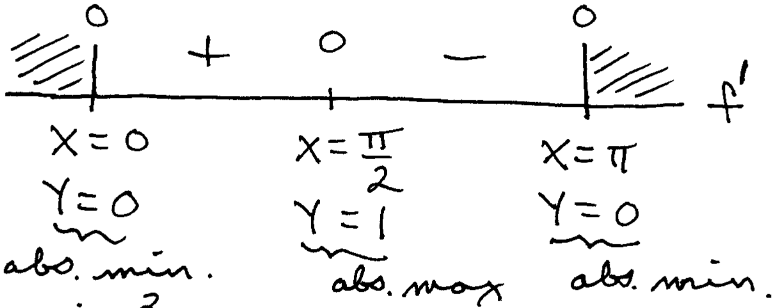
$+ (2 \cos x) \cdot \cos x$

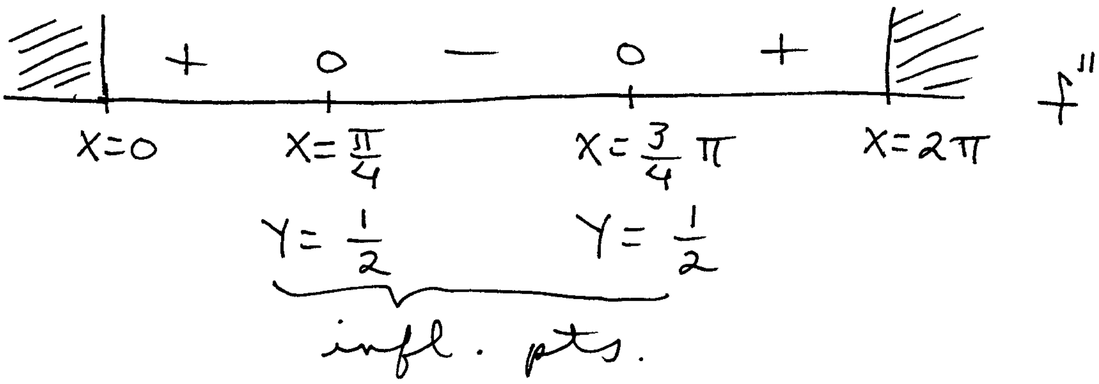
$= 2(\cos^2 x - \sin^2 x)$

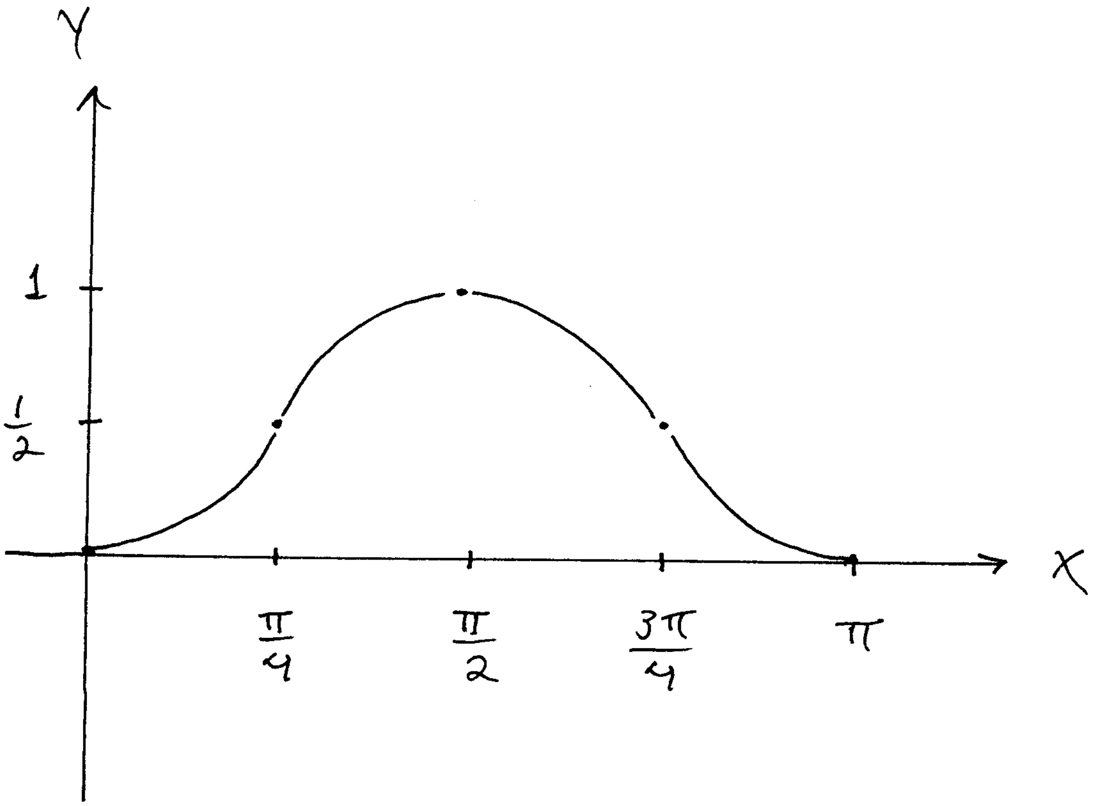
$= 0 \longrightarrow \cos^2 x = \sin^2 x \rightarrow$

$\sqrt{\cos^2 x} = \sqrt{\sin^2 x} \rightarrow |\cos x| = |\sin x| \rightarrow$

$\cos x = \pm \sin x \longrightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$




+
0
−
0
+
f''
X=0
$X=\frac{\pi}{4}$
$X=\frac{3}{4}\pi$
X=2π
$Y=\frac{1}{2}$
$Y=\frac{1}{2}$
infl. pts.


Y
1
$\frac{1}{2}$
X
$\frac{\pi}{4}$
$\frac{\pi}{2}$
$\frac{3\pi}{4}$
π