Math 16A
Kouba
Functions– Review

DEFINITION : In an equation composed of x's and y's, variable y is a function of x if each admissible x-value has exactly one y-value.

NOTE : The graph of a function passes the *vertical line test* . That is, a vertical line passed through the graph will touch the graph in at most one point.

EXAMPLE : Assume that $xy - 3 = x^2 + 2y$. Then $xy - 2y = x^2 + 3 \longrightarrow$

$$(x-2)y = x^2 + 3 \longrightarrow$$

$$y = \frac{x^2+3}{x-2} \longrightarrow$$

y is a function of x.

EXAMPLE : Assume that $xy^2 - 1 = x + y$. If $x = 1$, then

$$y^2 - 1 = 1 + y \longrightarrow$$

$$y^2 - y - 2 = 0 \longrightarrow$$

$$(y-2)(y+1) = 0 \longrightarrow$$

$$y = 2 \text{ or } y = -1 \longrightarrow$$

$$x = 1 \text{ has TWO } y\text{-values} \longrightarrow$$

y is NOT a function of x .

NOTATION : If y is a function of x, then we write $y = f(x)$.

EXAMPLE : If $y = x^2 + x$, then y is a function of x and we write $f(x) = x^2 + x$; then

a.) $f(-2) = (-2)^2 + (-2) = 4 - 2 = 2$.

b.) $f(2x-1) = (2x-1)^2 + (2x-1) = 4x^2 - 4x + 1 + 2x - 1 = 4x^2 - 2x$.

DEFINITION : Assume that $y = f(x)$ is a function. The domain of function f is the set of all admissible x-values. The range of function f is the set of all corresponding y-values.

EXAMPLE : Consider function $f(x) = \sqrt{2x-6}$. Then $2x - 6 \geq 0 \longrightarrow 2x \geq 6 \longrightarrow x \geq 3 \longrightarrow$

$$\text{DOMAIN} : x \geq 3.$$

Since $\sqrt{2x-6} \geq 0$, $f(3) = 0$, and $2x - 6$ gets infinitely large as x gets infinitely large, it follows that

$$\text{RANGE} : y \geq 0.$$

DEFINITION : A *function* $y = f(x)$ is one-to-one if each y-value has exactly one x-value. More precisely, a one-to-one function has the property that if $f(x_1) = f(x_2)$ (y-values are equal), then $x_1 = x_2$ (x-values are equal).

NOTE : The graph of a one-to-one function passes the *horizontal line test.* That is, a horizontal line passed through the graph will touch the graph in at most one point.

EXAMPLE : Consider the function (parabola) $y = x^2 - 5$. If $y = 4$, then

$$4 = x^2 - 5 \longrightarrow$$

$$x^2 = 9 \longrightarrow$$

$$x = 3 \text{ or } x = -3 \longrightarrow$$

$$y = 4 \text{ has TWO } x\text{-values} \longrightarrow$$

function y is NOT one-to-one .

<u>EXAMPLE</u> : Consider the function $f(x) = \dfrac{x}{x+3}$. Prove that f is one-to-one:

$$f(x_1) = f(x_2) \quad \longrightarrow$$

$$\frac{x_1}{x_1+3} = \frac{x_2}{x_2+3} \quad \longrightarrow$$

$$x_1(x_2+3) = x_2(x_1+3) \quad \longrightarrow$$

$$x_1x_2 + 3x_1 = x_1x_2 + 3x_2 \quad \longrightarrow$$

$$3x_1 = 3x_2 \quad \longrightarrow$$

$$x_1 = x_2 \quad \longrightarrow$$

function f IS one-to-one .

<u>DEFINITION</u> : Assume that $y = f(x)$ and $y = g(x)$ are functions. The <u>composition</u> of functions f and g is

$$(f \circ g)(x) = f(g(x)) \ .$$

<u>EXAMPLE</u> : Consider the functions $f(x) = \dfrac{x}{10-x}$ and $g(x) = \dfrac{1}{x+8}$. Then

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x+8}\right)$$

$$= \frac{\dfrac{1}{x+8}}{10 - \left(\dfrac{1}{x+8}\right)}$$

$$= \frac{\frac{1}{x+8}}{10 - \left(\frac{1}{x+8}\right)} \cdot \frac{x+8}{x+8}$$

$$= \frac{1}{10(x+8) - 1}$$

$$= \frac{1}{10x + 79} .$$

<u>DEFINITION</u> : The <u>inverse function</u> of function $y = f(x)$ is the function $y = f^{-1}(x)$ for which

$$f(f^{-1}(x)) = x \text{ and } f^{-1}(f(x)) = x .$$

<u>FACT</u> : If $y = f(x)$ is a one-to-one function, then f has an inverse function.

SEE INVERSE FUNCTION HANDOUT.

<u>EXAMPLE</u> : The function $f(x) = \frac{x}{x+3}$ is one-to-one. Find its inverse :

$$y = \frac{x}{x+3} \quad \longrightarrow \quad \text{(Switch variables.)} \quad x = \frac{y}{y+3}$$

$$\text{(Solve for } y.\text{)} \quad x(y+3) = y \quad \longrightarrow$$

$$xy + 3x = y \quad \longrightarrow$$

$$xy - y = -3x \quad \longrightarrow$$

$$y(x-1) = -3x \quad \longrightarrow$$

$$y = \frac{-3x}{x-1} \quad \longrightarrow \quad \text{inverse function is} \quad f^{-1}(x) = \frac{3x}{1-x} .$$