Math 16A
Kouba
Final Exam

KEY

______________________________ Print your name here.

__________________ Your exam ID #

1.) PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2.) No notes, books, or classmates may be used as resources for this exam. It is a violation of the University honor code to, in any way, assist another person in the completion of this exam. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted by it Thank you for your cooperation.

3.) Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will not receive full credit. The best way to get maximum partial credit is to write neatly and be organized.

4.) Make sure that you have ten (10) pages, including the cover page.

5.) You will be graded on proper use of limit notation.

6.) Include units on answers where units are appropriate.

7.) Use sign charts on all max./min. problems.

8.) You have until exactly 3:30 p.m. to complete the exam. Anyone who continues working on the exam after time is called risks being assessed a 20 point deduction on their exam score.

1.) (6 pts. each) Evaluate the following limits.

a.) $\lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to 2} \frac{2-x}{2x} \cdot \frac{1}{x-2} = \lim_{x \to 2} \frac{-1}{2x} = \frac{-1}{4}$

b.) $\lim_{x \to 1} \frac{x^2 - 1}{\sqrt{x} - 1} \overset{\frac{0}{0}}{=} \lim_{x \to 1} \frac{(x-1)(x+1)}{\sqrt{x} - 1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1}$

$= \lim_{x \to 1} \frac{(\cancel{x-1})(x+1)(\sqrt{x}+1)}{\cancel{x-1}} = 4$

c.) $\lim_{x \to 2^-} \frac{x^3 + 1}{x^2 - 4} = \text{"}\frac{9}{0^-}\text{"} = -\infty$

2.) (5 pts. each) Differentiate. Do not simplify answers.

a.) $y = \frac{\sqrt{5 + 2x}}{\cos(3x + 1)}$

$$y' = \frac{\cos(3x+1) \cdot \frac{1}{2}(5+2x)^{-1/2} \cdot 2 - \sqrt{5+2x} \cdot -\sin(3x+1) \cdot 3}{\cos^2(3x+1)}$$

b.) $y = \tan^3(\sin(5x))$

$$y' = 3\tan^2(\sin(5x)) \cdot \sec^2(\sin(5x)) \cdot \cos(5x) \cdot 5$$

3.) A baseball is hit straight upward. Its height above the ground after t seconds is $h(t) = 128\,t - 16\,t^2$ feet.

a.) (4 pts.) How long is the ball in the air ?

hit ground : $0 = h(t) \to 0 = 128t - 16t^2 = 16t(8-t)$

$\to \quad t = 8$ sec.

b.) (4 pts.) What is the ball's maximum height ?

highest pt. : $v(t) = 0 \to 128 - 32t = 0 \to$

$t = 4$ sec. $\to h(4) = 256$ ft.

c.) (4 pts.) What is the ball's velocity as it strikes the ground ?

$v(t) = 128 - 32t$ so

$v(8) = 128 - 32(8) = -128$ ft./sec.

d.) (4 pts.) What is the ball's acceleration as it strikes the ground ?

$a(t) = -32$ so

$a(8) = -32$ ft./sec.^{2}

4.) (12 pts.) Function $g(x) = 3 + \dfrac{x}{2+x}$. Find another function $f(x)$ so that $g(\,f(x)\,) = \sin x$. This problem is not difficult. Just do it.

$$3 + \frac{f(x)}{2+f(x)} = \sin x \to \frac{f}{2+f} = \sin x - 3 \to$$

$$f = (2+f)(\sin x - 3) \to$$

$$f = 2\sin x - 6 + f\sin x - 3f \to$$

$$4f - f\sin x = 2\sin x - 6 \to$$

$$f(4 - \sin x) = 2\sin x - 6 \to$$

$$f(x) = \frac{2\sin x - 6}{4 - \sin x}$$

5.) (20 pts.) For the following function f state its domain and determine all absolute and relative maximum and minimum values, inflection points, and intercepts. State clearly the x-values for which f is increasing, decreasing, concave up, and concave down. Neatly sketch the graph of f.

$f(x) = x\,(x-6)^2$

domain : all x-values

$f'(x) = x \cdot 2(x-6) + (x-6)^2 = (x-6)(2x+x-6)$

$= (x-6)(3x-6) = 0$

$+ \quad 0 \quad - \quad 0 \quad +$ f'

$X=2$, $Y=32$ rel. max. $X=6$, $Y=0$ rel. min.

$f''(x) = (x-6)(3) + (1)(3x-6)$

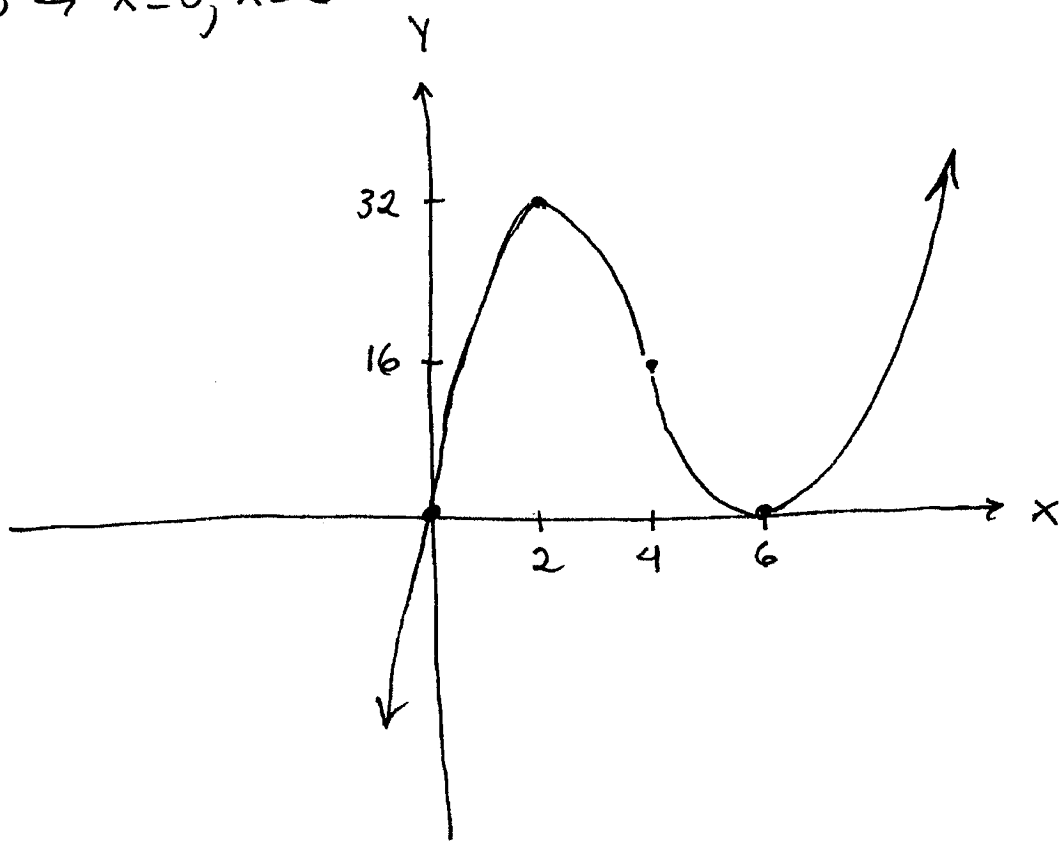
$= 6x - 24 = 0$

$- \quad 0 \quad +$ f''

$X=4$, $Y=16$ infl. pt.

f is ↑ for $x<2$, $x>6$,
f is ↓ for $2<x<6$,
f is ∪ for $x>4$,
f is ∩ for $x<4$;

$X=0 \rightarrow Y=0$
$Y=0 \rightarrow X=0,\ X=6$



6.) (10 pts.) Find two numbers x and y so that $x + y = 10$ and the sum $S = x^2 + 4y^2$ is a minimum. Your answer should include x, y, and the minimum S.

$$S = x^2 + 4(10-x)^2 \rightarrow S' = 2x - 8(10-x) = 10x - 80 = 0$$

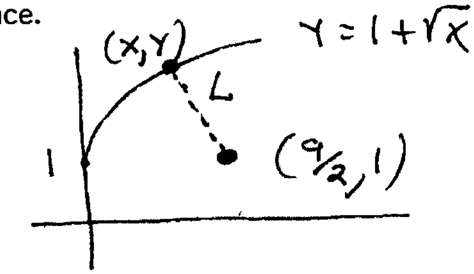
$-$ 0 $+$ S'

$x = 8$

$Y = 2$

min. $S = 80$

7.) (10 pts.) Find the point (x, y) on the graph of $y = 1 + \sqrt{x}$ which is nearest the point (9/2, 1). Your answer should include x, y, and the minimum distance.



min. $L = \sqrt{(x - \frac{9}{2})^2 + (Y-1)^2}$

and $Y = 1 + \sqrt{x}$ so

$$L = \sqrt{(x - \tfrac{9}{2})^2 + x} \longrightarrow$$

$$L' = \frac{1}{2}(\sim)^{-1/2}\left(2(x - \tfrac{9}{2}) + 1\right) = 0$$

$\rightarrow 2x - 8 = 0$

$-$ 0 $+$ L'

$x = 4$

$Y = 3$

and min. $L = \sqrt{\frac{17}{4}}$

8.) (10 pts.) There are 30 pear trees in an orchard. Each tree produces 500 pounds of pears. For each additional three (3) trees which are planted in the orchard, the output per tree drops by 25 pounds. How many pear trees should be added to the existing orchard in order to have the largest total output (pounds) of pears ? Your answer should use calculus and include the number of trees, the output per tree, and the maximum total output.

Let x : # of groups of 3 trees,

max. output $T = $ (# of trees)(output per tree)

$$\rightarrow T = (30 + 3x)(500 - 25x)$$

$$\rightarrow T' = (30 + 3x)(-25) + (3)(500 - 25x)$$

$$= -750 - 75x + 1500 - 75x = 750 - 150x = 0 \rightarrow$$

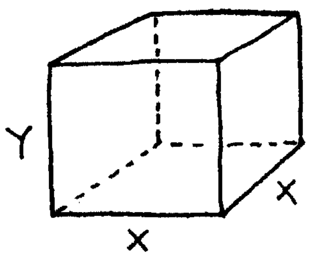
$+$ 0 $-$ T'

$x = 5$ so $30 + 5(3) = 45$ trees

with $500 - 25(5) = 375$ ~~pounds~~ lbs. per tree

and max. $T = (45)(375) = 16{,}875$ lbs.

9.) (10 pts.) A closed rectangular box with a square base is to be constructed from two different materials. Material for the top and bottom costs \$3/ft.^{2} and material for the sides costs \$2/ft.^{2}. Find the dimensions of the box of largest volume which can be constructed for exactly \$72. Your answer should include the length, width, height, and volume of the box.



$3(2x^2) + 2(4xY) = 72 \rightarrow$

$6x^2 + 8xY = 72 \rightarrow 3x^2 + 4xY = 36 \rightarrow$

$Y = \dfrac{36 - 3x^2}{4x}$ and

max. volume $V = x^2 Y \rightarrow$

$V = x^{\cancel{2}} \dfrac{(36 - 3x^2)}{4\cancel{x}} = 9x - \frac{3}{4}x^3 \rightarrow V' = 9 - \frac{9}{4}x^2 = 0 \rightarrow$

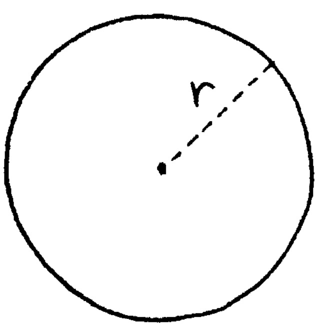
$+ \quad 0 \quad -$ V'

$x = 2$ ft.

$Y = 3$ ft.

and max. $V = 12$ ft.^{3}

10.) (12 pts.) The circumference of a circle is increasing at the rate of 2 ft./sec. At what rate is the area of the circle changing when the radius of the circle is 6 ft. ?



$C = 2\pi r$ and $A = \pi r^2$,

$\dfrac{dC}{dt} = 2$ ft/sec., find $\dfrac{dA}{dt}$ when $r = 6$ ft. $\rightarrow$

$\dfrac{dC}{dt} = 2\pi \dfrac{dr}{dt} \rightarrow 2 = 2\pi \dfrac{dr}{dt}$ so

$\dfrac{dr}{dt} = \dfrac{1}{\pi}$ ft/sec. ; $A = \pi r^2 \xrightarrow{D_t}$

$\dfrac{dA}{dt} = 2\pi r \cdot \dfrac{dr}{dt} = 2\cancel{\pi}(6)\left(\dfrac{1}{\cancel{\pi}}\right) = 12$ ft.^{2}/sec.

11.) (12 pts.) Use limits to determine all horizontal asymptotes for
$y = \dfrac{x}{\sqrt{x^2+25}}$

$$\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+25}} = \lim_{x\to+\infty} \frac{x}{\sqrt{x^2\left(1+\frac{25}{x^2}\right)}} = \lim_{\substack{x\to+\infty \\ x>0}} \frac{x}{|x|\sqrt{1+\frac{25}{x^2}}}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}\sqrt{1+\frac{25}{x^2}}} = \frac{1}{\sqrt{1+0}} = 1 \text{ so } \boxed{y=1} \text{ is H.A.}$$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+25}} = \lim_{\substack{x\to-\infty \\ x<0}} \frac{x}{|x|\sqrt{1+\frac{25}{x^2}}} = \lim_{x\to-\infty} \frac{\cancel{x}}{-\cancel{x}\sqrt{1+\frac{25}{x^2}}}$$

$$= \frac{1}{-\sqrt{1+0}} = -1 \text{ so } \boxed{y=-1} \text{ is H.A.}$$

12.) (12 pts.) A tank contains 100 gallons of solution in which 4 lbs. of salt is dissolved. A solution containing 3/4 lbs. of salt per gallon begins flowing into the tank at the rate of 8 gal./min. After how many minutes t will the concentration of salt in the tank be 2/3 lbs./gal. ?

Initially, 100 gal., 4 lbs. of salt

New solution: $\frac{3}{4}\,\frac{\text{lb.}}{\text{gal.}}$, $8\,\frac{\text{gal.}}{\text{min.}}$, $6\,\frac{\text{lbs.}}{\text{min.}}$

after t min. concentration is (lbs./gal.)

$$\frac{2}{3} = \frac{4+6t}{100+8t} \rightarrow 200+16t = 12+18t \rightarrow$$

$$188 = 2t \rightarrow t = 94 \text{ min.}$$

13.)(12 pts.) Assume that $x \sec y = y^2 + xy$. Find $y' = dy/dx$.

$$\xrightarrow{D} x \cdot \sec y \tan y \cdot y' + 1 \cdot \sec y = 2yy' + xy' + y$$

$$\rightarrow x \sec y \tan y \cdot y' - 2yy' - xy' = y - \sec y$$

$$\rightarrow y'(x \sec y \tan y - 2y - x) = y - \sec y$$

$$\rightarrow y' = \frac{y - \sec y}{x \sec y \tan y - 2y - x}$$

14.) (12 pts.) Use $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ to differentiate the function $f(x) = 3x - x^2$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{3(x+\Delta x) - (x+\Delta x)^2 - [3x - x^2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{3x} + 3 \cdot \Delta x - \cancel{x^2} - 2x \cdot \Delta x - (\Delta x)^2 - \cancel{3x} + \cancel{x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x} \cdot [3 - 2x - \Delta x]}{\cancel{\Delta x}}$$

$$= 3 - 2x$$

15.) (12 pts.) Use differentials to estimate the value of $\sqrt{1.2} + (1.2)^{1/3}$.

Let $f(x) = \sqrt{x} + x^{1/3}$ and $x: 1 \to 1.2$ so $\Delta x = 0.2$

$$f'(x) = \frac{1}{2}x^{-1/2} + \frac{1}{3}x^{-2/3} = \frac{1}{2\sqrt{x}} + \frac{1}{3x^{2/3}},$$

$$\Delta Y = f(1.2) - f(1) = \sqrt{1.2} + (1.2)^{1/3} - 2$$

$$dY = f'(1) \cdot \Delta x = \left(\frac{1}{2} + \frac{1}{3}\right) \cdot (0.2) = \frac{5}{6} \cdot \frac{1}{5} = \frac{1}{6} \approx 0.167$$

and $\Delta Y \approx dY$ so

$$\sqrt{1.2} + (1.2)^{1/3} - 2 \approx 0.167$$ or

$$\sqrt{1.2} + (1.2)^{1/3} \approx 2.167$$

16.) (12 pts.) Assume that the maximum absolute percentage error in measuring the volume of a sphere is 27%. Use differentials to determine the resulting maximum absolute percentage error in computing the radius of the sphere ? (The volume of a sphere is $V = (4/3)\,\pi\, r^3$.)

$\frac{|\Delta V|}{V} \le 27\%$, find $\frac{|\Delta r|}{r}$:

$$\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{|V' \cdot \Delta r|}{V} = \frac{|\cancel{4}\cancel{\pi}\cancel{r^2} \cdot \Delta r|}{\frac{\cancel{4}}{3}\cancel{\pi}r^{\cancel{3}}}$$

$$= 3\,\frac{|\Delta r|}{r} \le 27\%$$ so

$$\frac{|\Delta r|}{r} \le 9\%$$

Extra Credit Problems-- Each of the following problems is optional and each is worth 10 points.

1.) Consider the function f whose derivative f' is given in the graph below. I repeat, the graph below is that of f', NOT f. However, answer the following questions about f, NOT f'.

a.) List the x-value(s) for which f has a relative maximum.

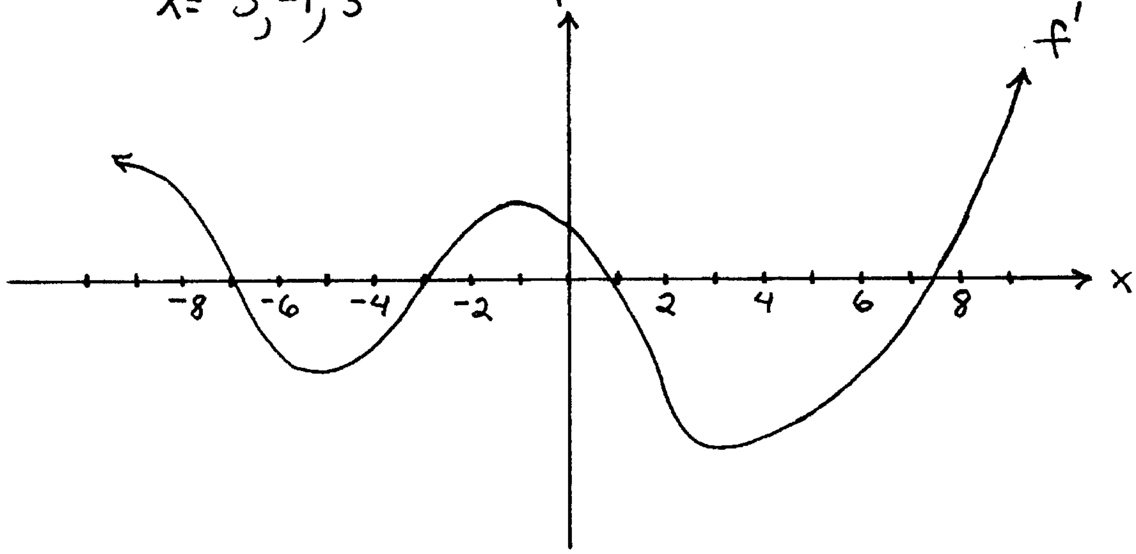
$x = -7, 1$

b.) List the x-value(s) for which f has a relative minimum.

$x = -3, 7.5$

c.) List the x-value(s) for which f has an inflection point.

$x = -5, -1, 3$



2.) A square is inscribed in the given right triangle. Find the area of the square.

By similar triangles

$\frac{12}{5} = \frac{12-x}{x} \rightarrow$

$12x = 60 - 5x \rightarrow$

$17x = 60 \rightarrow$

$x = \frac{60}{17}$ so

area $A = \left(\frac{60}{17}\right)^2$