Homework Problems

① Recall $\underset{\sim}{x}$ and $\underset{\sim}{y}$ denote the coordinates maps

$$\underset{\sim}{x} : M \to \mathbb{R}^4 \qquad \underset{\sim}{y} : M \to \mathbb{R}^4$$
$$p \mapsto \underset{\sim}{x}(p) \qquad p \mapsto \underset{\sim}{y}(p)$$

as well as points in $\mathbb{R}^4$ $\underset{\sim}{x} \in \mathbb{R}^4$, $\underset{\sim}{y} \in \mathbb{R}^4$.

Let $J = \frac{\partial y}{\partial x}$ denote the 4×4 matrix $\frac{\partial y^\alpha}{\partial x^i}$ ← row (α), colm (i)

which is really $\frac{\partial}{\partial x^i} \underbrace{\left(y^\alpha \circ \underset{\sim}{x}^{-1} \right)}_{\substack{\mathbb{R}^4 \to \mathbb{R} \\ \underset{\sim}{x} \to y^\alpha}}$

Let $\bar{J} = \frac{\partial x}{\partial y}$.

Now $\bar{J} = J^{-1}$ follows because

$$\underset{\sim}{y} = \underbrace{(\underset{\sim}{y} \circ \underset{\sim}{x}^{-1})}_{\substack{\mathbb{R}^4 \leftarrow \mathbb{R}^4 \\ \underset{\sim}{y} \leftarrow\!\shortmid \underset{\sim}{x}}} \circ \underbrace{(\underset{\sim}{x} \circ \underset{\sim}{y}^{-1})}_{\substack{\mathbb{R}^4 \leftarrow \mathbb{R}^4 \\ \underset{\sim}{x} \leftarrow\!\shortmid \underset{\sim}{y}}} \circ (\underset{\sim}{y}) \qquad (*)$$

← pt in $\mathbb{R}^4$

is the identity map $\underset{\sim}{y} \mapsto \underset{\sim}{y}$.

Differentiaty (*) using the chain rule on the two parts gives (sum i from 0 to 4)

$$\underbrace{\frac{\partial y^\alpha}{\partial y^\beta}}_{\delta^\alpha_\beta} = \underbrace{\frac{\partial y^\alpha}{\partial x^i}}_{J} \underbrace{\frac{\partial x^i}{\partial y^\beta}}_{\bar{J}} \iff I = J \cdot \bar{J}$$

$$\Rightarrow \bar{J} = J^{-1}$$

HW. Work this out for 2-d spacetimes

$i, j = 0, 1 \qquad \alpha, \beta = 0, 1.$

(2) Assume a 2-d spacetime. Show that

$$a^i \frac{\partial}{\partial x^i} (f \circ \underset{\sim}{x}^{-1}) = b^\alpha \frac{\partial}{\partial y^\alpha} (f \circ y^{-1})$$

so long as $a^i = \frac{\partial x^i}{\partial y^\alpha} b^\alpha$ & $\frac{\partial}{\partial x^i} = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha}$

(Hint: write everything out by expanding summation of up-down indices)