Homework
Math 180: Introduction to GR
Temple-Winter 2018

(3) Summarize the article: https://www.ucdavis.edu/news/doing-without-dark-energy/

(4) Assume only the transformation laws for vectors. Let $X_P = a^i \frac{\partial}{\partial x^i} = a^\alpha \frac{\partial}{\partial y^\alpha} = Y_P$, and assume f is a scalar function on the spacetime manifold $\mathcal{M}$. Let $f \circ \mathbf{x}^{-1}$ and $f \circ \mathbf{y}^{-1}$ be the x- and y-coordinate representations of f, respectively. Prove that

$$a^i \frac{\partial}{\partial x^i} f \circ \mathbf{x}^{-1} = a^\alpha \frac{\partial}{\partial y^\alpha} f \circ \mathbf{y}^{-1}.$$

Explain why this makes sense as functions from coordinates to coordinates even though f is defined on spacetime, and you can't "differentiate spacetime" directly.

Let f be a function on spacetime, $f : \mathcal{M} \to \mathbf{R}$. Let $X_P = a^i \frac{\partial}{\partial x^i}$ be a vector at $\mathbf{x}(P)$, and let $df = \frac{\partial f}{\partial x^i} dx^i$ be the 1-form which corresponds to the differential of f. Explain how X operates on f to give the gradient of f in direction X: And then explain how, alternatively, we can view df as operating on X to give the gradient of f in direction X.

Finally, say in words why this gives an interpretation of a vector as something independent of coordinates.

(5) Show that if g_{ij} are the components of a gravitational metric in x-coordinates, and g_{ij} transforms like a $(0, 2)$ tensor over to

y-coordinates as $\bar{g}_{\alpha\beta}$, that is,

$$\bar{g}_{\alpha\beta} = \frac{\partial x^i}{\partial y^\alpha} g_{ij} \frac{\partial x^j}{\partial y^\beta},$$

then g_{ij} is symmetric ($g_{ij} = g_{ji}$) if and only if $\bar{g}_{\alpha\beta}$ is symmetric, and g_{ij}, as a 4×4 matrix, has an inverse if and only if $\bar{g}_{\alpha\beta}$ has an inverse as a 4×4 matrix. (Hint: Write the tensor transformation laws as matrix multiplication, and use properties of matrices, like "A matrix is invertible if and only if its determinant is non-zero." "The product of nonsingular matrices is nonsingular", etc.)

(6) (I) Let g and J be 3×3 matrices, and let $gJ = A$. Write out the rows of all three matrices, and explain matrix multiplication by the "rows of G contract with the rows of J to create the rows of A". Then write out the columns of all three matrices, and explain how "the columns of J contract with the columns of G to create the columns of A".

(II) Now let g_{ij} and J^i_j be $(0,2)$ and $(1,1)$ tensors viewed as 3×3 matrices, i the row and j the column, and consider the two matrix multiplications $g_{ik}J^k_j = A_{ij}$ and $g_{kj}J^k_i = A_{ij}$ (the latter being equivalent to $J^k_i g_{kj} = A_{ij}$ because the order in which you list the tensors doesn't matter when you use the summation convention to express matrix multiplication) .

(a) Using the column interpretation of matrices, explain why $g_{ik}J^k_j = A_{ij}$ expresses matrix multiplication $gJ = A$.

(b) Using the row interpretation of matrices, explain why $g_{kj}J^k_i = A_{ij}$ expresses matrix multiplication $J^t g = A$.

(7) Prove that if S^i_j and T^i_j are the components of a $(1,1)$-tensors in x-coordinates, then $A^i_j = S^i_k T^k_j$ (sum repeated up-down indices from 0 to 3) transforms like a $(1,1)$-tensor. What would the general theorem about tensors arbitrary tensors S, T be?

(8) Let T^i_j be the components of a $(1,1)$-tensor T at a point in x-coordinates, and assume there is a vector $X = a^i \frac{\partial}{\partial x^i}$ such that $T^i_j a^j = \lambda a^i$. (That is, λ is an eigenvalue of the 4×4 matrix T^i_j.) Prove that λ is independent of coordinates. (Hint: See what's true in y-coordinates, for any other coordinate system y.)

(9) Let X, Y, Z be three independent vectors in $T_P(M)$ (the tangent space of M at P) given in x-coordinates by $X = a^i \frac{\partial}{\partial x^i}$, $Y = b^j \frac{\partial}{\partial x^j}, Z = c^k \frac{\partial}{\partial x^k}$. Let $\mathbf{n} = (n_0, n_1, n_2, n_3)$ be the x-coordinate unit normal to the hyperplane spanned by X, Y, Z, so that $\mathbf{n} \cdot \mathbf{a} = \mathbf{n} \cdot \mathbf{b} = \mathbf{n} \cdot \mathbf{c} = 0$, and $\mathbf{n} \cdot \mathbf{n} = 1$, where *dot* is the dot product in x-coordinates.

(a) Show that if we assume n_i transform co-variantly as a down index to $\bar{\mathbf{n}} = (\bar{n}_0, \bar{n}_1, \bar{n}_2, \bar{n}_3)$ in y-coordinates, then

$$\bar{\mathbf{n}} \cdot \bar{\mathbf{a}} = \bar{\mathbf{n}} \cdot \bar{\mathbf{b}} = \bar{\mathbf{n}} \cdot \bar{\mathbf{c}} = 0.$$

(b) Is $\bar{n} \cdot \bar{n} = 1$ in y-coordinates? Explain.

(10) Lete $\{X_0, X_1\}$ be a positively oriented orthonormal frame in $1+1$ special relativity. Define the $\bar{x}$ coordinate system in terms of the given x-coordinate system by specifying that P has

x-coordinates (x^0, x^1) and $\bar{x}$-coordinates $(\bar{x}^0, \bar{x}^1)$ if and only if

$$x^0 \frac{\partial}{\partial x^0} + x^1 \frac{\partial}{\partial x^1} = \bar{x}^0 X_0 + \bar{x}^1 X_1.$$

Argue that in this case,

$$X_0 = \frac{\partial}{\partial \bar{x}^0}, \quad X_1 = \frac{\partial}{\partial \bar{x}^1}.$$

(Hint: What defines $\frac{\partial}{\partial \bar{x}^i}$ in the first place?)

(11) In the field of *asymptotics*, we say "$f(t)$ is $O(t^n)$ as $t \to 0$" to mean that there exists a constant $C > 0$ such that $|f(t)| \leq C|t^n|$ for t sufficiently small. (Often we omit to add "as $t \to 0$", but this is always implied.)

(a) Use Taylor's theorem to show that:

$$\frac{1}{1+x} = 1 - x + O(x^2), \quad \frac{1}{1-x} = 1 + x + O(x^2),$$

and

$$\sqrt{1+x} = 1 + \frac{1}{2}x + O(x^2), \quad \sqrt{1-x} = 1 - \frac{1}{2}x + O(x^2).$$

(b) Use these to show that

$$\sqrt{1 - \left(\frac{v}{c}\right)^2} = 1 - \frac{1}{2}\left(\frac{v}{c}\right)^2 + O\left(\frac{v}{c}\right)^4,$$

and

$$\frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + O\left(\frac{v}{c}\right)^4.$$

(12) Twin Paradox: Imagine two twin observers who start out fixed with respect to a given Minkowski coordinate system. One twin remains at rest in that frame, while the other goes off in a rocket ship and returns to see the first twin at a later time. According to our time dilation, the second traveling twin will age less than the twin who remains at rest in the Minkowski frame, because "moving observers appear to age slower". The Paradox is, to the second twin, the first twin appears to have gone off on a travel and returned to him, so by symmetry, why wouldn't the second twin be older than the first, a contradiction? Resolve the paradox in words.

(13) Show $L(\theta)L(\bar{\theta}) = L(\theta + \bar{\theta})$, and use this to prove the relativistic addition of velocities formula

$$\bar{\bar{v}} = \frac{v + \bar{v}}{1 + \frac{v\bar{v}}{c^2}},$$

where v is the velocity of the $\bar{x}$-frame with respect to the unbarred frame, and $\bar{v}$ is the velocity of a third $\bar{\bar{x}}$-frame with respect to the barred frame.

(14) Assume X is non-lightlike, so $< X, X > \neq 0$. Derive the relativistic version of the "orthogonal projection of a vector Y onto a vector X" given by

$$Proj_X Y = \frac{< X, Y >}{< X, X >} X,$$

and interpret it geometrically. (Hint: Start with X, write Y as a linear combination of X and the unit vector othogonal to X, and use the inner product to solve for the components.)