Homework-3
Math 180: Introduction to GR
Temple-Winter 2018

(1) Consider the Lorentz transformation representing a canonical boost:

$$\begin{pmatrix} \gamma & -\gamma\frac{\mathbf{v}^t}{c} \\ -\gamma\frac{\mathbf{v}}{c} & I+(\gamma-1)\frac{\mathbf{v}\mathbf{v}^t}{|\mathbf{v}|^2}. \end{pmatrix} \tag{1}$$

where $\mathbf{v}$ is the 3-velocity of the moving frame $\bar{x}$ as measured in the fixed frame x, and

$$\gamma = \frac{1}{\sqrt{1-\frac{|\mathbf{v}|^2}{c^2}}}.$$

Now the vector $\mathbf{X}_0 = \frac{\partial}{\partial x^0}$ with components $(1,0,0,0)$, is the unit timelike vector in x-coordinates. Viewing (1) as mapping components of vectors to components of vectors (identifying vectors with points in the coordinate system), and assuming the image of $(1,0,0,0)$ gives the components of the unit timelike vector $\frac{\partial}{\partial \bar{x}^0}$ in the x-coordinates, find the components of $\frac{\partial}{\partial \bar{x}^0}$ in x-coordinates, and use $x^0 = ct$ to show that it moves at velocity $\mathbf{v}$ relative to the observer fixed in x-coordinates.

(2) Let $\mathbf{y} = (y^1, ..., y^n) \in \mathbf{R}^n$, and consider the autonomous system of ODE's

$$\dot{\mathbf{y}} = f(\mathbf{y}), \tag{2}$$

with initial condition

$$\mathbf{y}(t_0) = \mathbf{y}_0, \tag{3}$$

where $f(\mathbf{y}) = (f^1(\mathbf{y}), ..., f^n(\mathbf{y}))$ is assumed to be Lipschitz continuous, so by our Nonlinear ODE Theorem, we know that (2), (3) always has a unique solution. (I use up indices here only because we apply this to ODE's for components of vector fields, which are up, but for this theorem they could be up or down as we work in one coordinate system.)

(**a**) Prove that if $\mathbf{y}(t)$ solves (2), then so does $\mathbf{y}(t+s)$ for any constant s.

(**b**) Given a solution $\mathbf{y}(t)$, define the *trajectory* of the solution to be the curve of $\mathbf{y}$ values in $\mathbf{R}^n$ obtained as the parameter t ranges over all its values. By this, together with (**a**), the trajectory of $\mathbf{y}(t)$ is the same as the trajectory of $\mathbf{y}(t+s)$ for any fixed s, because s just changes the parameterization of the trajectory by a translation of the input by s. Prove that no two trajectories of (2) intersect.
(Hint: Assume the trajectories of two solutions $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ intersect at some point $\mathbf{y}_0$, and use (**a**) together with the uniqueness of solutions to prove that in fact $\mathbf{y}_1(t)$ must equal $\mathbf{y}_2(t+s)$ for some translation s, and hence they represent the same trajectory.)

(3) Let Γ be a connection, with components Γ^i_{jk} in coordinate system x. Prove:

(**a**) Prove that $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ is a $(1,2)$ tensor (called the Torsion tensor), and use this to prove that *symmetry*, i.e., $\Gamma^i_{jk} = \Gamma^i_{kj}$, is a coordinate independent condition. (Hint: use the transformation law to prove that the difference is a tensor, and a tensor is determined by its values in any one coordinate system.)

(**b**) Use (**a**) to prove that a connection that is equal to zero in a locally inertial frames, must be symmetric.

(**c**) Similarly, prove that the difference between any two connections is a tensor.

(4) Verify properties (1)-(4) of the covariant derivative on page 23 of 9-The Spacetime Connection.

(5) Write out in your own words, the derivation of the covariant derivative for a 1-form on the bottom of page 26 in the notes on 9-The Spacetime Connection.

(6) Assume $R^k_{lij} Z^l$ transforms like a $(1, 2)$ tensor for every vector Z. Prove that R^k_{lij} transforms like a $(1, 3)$ tensor.