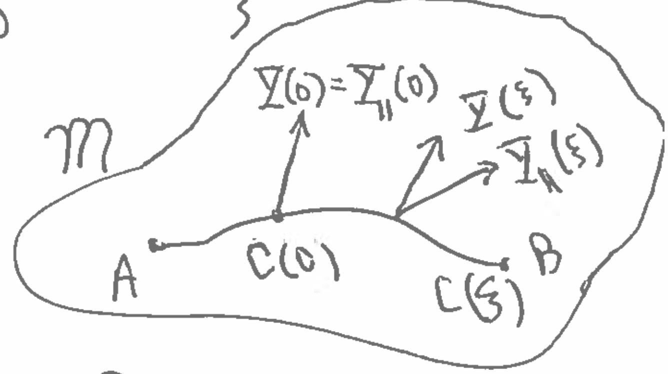
$\boxed{\text{Curvature}}$

10 ①

Curvature :

- We have $\nabla_X Y \Big|_p = \lim_{\xi \to 0} \dfrac{Y(\xi) - Y_{\parallel}(\xi)}{\xi}$



Defined invariantly :

$X \equiv$ vector field with integral curve $C(\xi)$, $C(0) = P$

$Y(\xi) \equiv Y(C(\xi))$ defined along $C(\xi)$

$Y_{\parallel}(\xi)$ = parallel translation of $Y(0)$ along $C(\xi)$

$$\frac{Y(\xi) - Y_{\parallel}(\xi)}{\xi} \in T_{C(\xi)} m$$

We know when vectors converge $\Leftrightarrow$ components converge in every coord system

$\Rightarrow$ limit is in $T_{C(0)} m$

Note : Integral curves of X autonomous ODE

$\Rightarrow$ free to $\xi \longrightarrow \xi + s$ any translation.

• In a coordinate system x :

$$(\nabla_{\mathbf{X}} \mathbf{Y})^i = \lim_{\xi \to 0} \frac{Y^i - Y^i_{\parallel}}{\xi}$$

$$= \frac{dY^i}{d\xi} - \frac{dY^i_{\parallel}}{d\xi} = \underbrace{\mathbf{X}(Y^i)}_{X^i \frac{\partial}{\partial x^i}(Y^i(\underset{\sim}{x}))} + \Gamma^i_{jk} Y^j X^k$$

Define :

$$\boxed{\begin{aligned} \nabla_{\mathbf{X}} f &= \mathbf{X}(f) = X^i \frac{\partial}{\partial x^i} f(\underset{\sim}{x}) \\ (\nabla_{\mathbf{X}} \omega)_i &= X^\sigma \frac{\partial \omega_i}{\partial x^\sigma} - \Gamma^\sigma_{i\tau} \omega_\sigma X^\tau \\ (\nabla_{\mathbf{X}} \mathbf{Y})^i &= X^\sigma \frac{\partial}{\partial x^\sigma} Y^i + \Gamma^\sigma_{i\tau} Y^i X^\tau \end{aligned}}$$

$$(\nabla_{\mathbf{X}} T)^i_{\ j} = X^\sigma \frac{\partial}{\partial x^\sigma} T^i_{\ j} + \Gamma^i_{\tau k} T^\tau_{\ j} X^k - \Gamma^\tau_{jk} T^i_{\ \tau} X^k$$

linear $\equiv (\nabla T)^i_{\ jk} = T^i_{\ j,k} + \Gamma^i_{\tau k} T^\tau_{\ j} - \Gamma^\tau_{jk} T^i_{\ \tau} = T^i_{\ j;k}$

<u>Notation</u>: $$Y^i_{;j} = \left(\nabla_{\frac{\partial}{\partial x^j}} \underline{Y}\right)^i$$

Then: $Y^i_{;j}$ are components of a $\binom{1}{1}$ tensor

$$\nabla Y = \underset{\text{x-components}}{\underset{\uparrow}{Y^i_{;j}}} \underbrace{dx^j \otimes \frac{\partial}{\partial x^i}}_{\text{x-basis vectors for } \mathcal{T}^1_1(p)}$$

$$Y^i_{;j} = Y^i_{,j} + \Gamma^i_{\sigma j} Y^\sigma \quad \Big/\!\!\Big/ \quad \nabla_{\underline{X}} \underline{Y} = X^i\left(\nabla_{\frac{\partial}{\partial x^i}} Y\right)^j = X^i Y^j_{;i}$$

<u>Problem</u>: We want a tensor R involving 2nd deriv's of g_{ij} which has the property that it measures the 2nd order errors

$$g_{ij}(p) = \eta_{ij}(p) + \underbrace{O\left(|\underset{\sim}{x} - \underset{\sim}{x}(p)|^2\right)}_{\text{Measured by } R^i_{jkl} \text{ a } \underline{\text{tensor}}}$$

Such that, if $R^i_{jk\ell} \equiv 0$, then $\exists$ global coord system in which

$$g_{ij} \equiv \eta_{ij} \quad \forall\, \underset{\sim}{x} \in \mathbb{R}^4 .$$

Note: Not enough that $R^i_{jk\ell}(P) = 0$, we need $R^i_{jk\ell}(\underline{x}) = 0$ in nbhd of P to get $g_{ij} = \eta_{ij}$ in nbhd of P.

- <u>Thm</u> ① $\left(\nabla_j \nabla_i Z^k - \nabla_i \nabla_j Z^k\right) = \underbrace{R^k_{\ell ij}}_{\text{Riemann Curvature Tensor}} Z^\ell$

We know $\nabla_j \nabla_i Z$ & $\nabla_i \nabla_j Z$ are tensors, but the dependence on derivatives of Z cancel out in the commutor $\Rightarrow R^k_{\ell ij}$ indept of Z!

$\cancel{R^k_{\ell ij} = \left\{ \Gamma^k_{\ell j,i} - \Gamma^k_{\ell i,j} + \Gamma^r_{\ell j} \Gamma^k_{ri} - \Gamma \right.}$

where

$$R^k_{\ell ij} = \underbrace{\Gamma^k_{\ell j,i} - \Gamma^k_{\ell i,j}}_{\text{"Curl"}} + \underbrace{\Gamma^\tau_{\ell j}\Gamma^k_{\tau i} - \Gamma^\tau_{\ell i}\Gamma^k_{\tau j}}_{\text{"Commutator"}}$$

Thm 2: $R^k_{\ell ij} \equiv 0$ in nbhd of P iff $\exists$ coord system in nbhd of P in which $g_{ij} = \eta_{ij}$.

Principle: "Commutators cancel out non-tensorial 2nd derivative"

<u>Defn</u>: Given two vector fields $\underline{X}$ & $\underline{Y}$,

$$[\underline{X},\underline{Y}] = \underbrace{\underline{X}(\underline{Y})}_{\text{non-tensorial}} - \underbrace{\underline{Y}(\underline{X})}_{\text{non-tensorial}}$$ is a vector field

$$[\underline{X},\underline{Y}] = [\underline{X},\underline{Y}]^i \frac{\partial}{\partial x^i}$$ transforms like $\binom{1}{0}$-tensor

Pf. Consider... $\underline{X} = X^i \frac{\partial}{\partial x^i}$ $\underline{Y} = Y^j \frac{\partial}{\partial x^j}$

$$\underline{X}(f) = X^i \frac{\partial}{\partial x^i}(f \circ x^{-1})(\underset{\sim}{x}) = X^i \frac{\partial f}{\partial x^i}(\underset{\sim}{x})$$ scalar

gives rate of change of f in direction $\underline{X}$

$$= \frac{d}{ds} f(\underset{\sim}{x} \circ c_x(s))$$

integral curve of $\underline{X}$ thru p

$$\underline{X}(\underline{Y})^k = X^i \frac{\partial}{\partial x^i} Y^k(\underset{\sim}{x})$$

<u>not a fn</u>: component of $\binom{1}{0}$-tensor

"Treat component $Y^k(\underset{\sim}{x})$ as if it were a scalar fn, even tho it isn't because it's a component"

To see its not a tensor, transform to $\underset{\sim}{y}$-coordinates: $\underline{X} = \bar{X}^\alpha \frac{\partial}{\partial y^\alpha}$, $\underline{Y} = \bar{Y}^\alpha \frac{\partial}{\partial y^\alpha}$

$$\underline{X}(\underline{Y})^k = \underbrace{X^i \frac{\partial}{\partial x^i}}\ \underbrace{Y^k(\underset{\sim}{x})}$$

Treating $Y^k(\underset{\sim}{x}) = f \circ \underset{\sim}{x}^{-1}$ as a fn so $X^i \frac{\partial}{\partial x^i} f \circ \underset{\sim}{x}^{-1} = \bar{X}^\alpha \frac{\partial}{\partial x^\alpha} f \circ \underset{\sim}{y}^{-1}$

$$\bar{X}^\alpha \frac{\partial}{\partial y^\alpha} \left[\frac{\partial x^k}{\partial y^\beta} \bar{Y}^\beta(\underset{\sim}{y}) \right]$$

$$= \bar{X}^\alpha \frac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta} \bar{Y}^\beta + \bar{X}^\alpha \frac{\partial x^k}{\partial y^\beta} \frac{\partial}{\partial y_\alpha} \bar{Y}^\beta$$

$$\underline{Y}(\underline{X})^k = \underbrace{\bar{Y}^\alpha \frac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta} \bar{Y}^\beta}_{\text{nontensorial and deriv term}} + \underbrace{\bar{Y}^\alpha \frac{\partial x^k}{\partial y^\beta} \frac{\partial}{\partial y^\alpha} \bar{X}^\beta}_{\text{tensorial}}$$

$$[\underline{X}, \underline{Y}]^k = \bar{X}^\alpha \frac{\partial x^k}{\partial y^\beta} \frac{\partial}{\partial y^\alpha} \bar{Y}^\beta - \bar{Y}^\alpha \frac{\partial x^k}{\partial y^\beta} \frac{\partial}{\partial y^\alpha} \bar{X}^\beta$$

$$= \frac{\partial x^k}{\partial y^\beta} \left\{ \bar{X}^\alpha \frac{\partial}{\partial y^\alpha} \bar{Y}^\beta - \bar{Y}^\alpha \frac{\partial}{\partial y^\alpha} \bar{X}^\beta \right\}$$

$$= \frac{\partial x^k}{\partial y^\beta} [\underline{X}, \underline{Y}]^\beta \Rightarrow [X, Y]^k \text{ is } \binom{1}{0}\text{-tensor}$$

Similarly for Curvature –

$$\left(\nabla_j \nabla_i Z - \nabla_i \nabla_j Z\right)^k = Z^k_{;i;j} - Z^k_{;j;i}$$

$$Z^k_{;i;j} = \left(Z^k_{;i}\right)_{;j} = \left(Z^k_{,i} + \Gamma^k_{i\sigma} Z^\sigma\right)_{;j}$$

$$= Z^k_{,ij} + \Gamma^k_{i\sigma,j} Z^\sigma + \Gamma^k_{i\sigma} Z^\sigma_{,j} + \boxed{\Gamma^k_{\tau j} Z^\tau_{,i}} + \Gamma^k_{\tau j}\Gamma^\tau_{i\sigma} Z^\sigma$$

$$- \cancel{\Gamma^\tau_{ij} Z^k_{,\tau}} - \Gamma^\tau_{ij}\Gamma^k_{\tau\sigma} Z^\sigma$$

$$Z^k_{;j;i} = \left(Z^k_{;j}\right)_{;i} = \left(Z^k_{,j} + \Gamma^k_{j\sigma} Z^\sigma\right)_{;i}$$

$$= Z^k_{,ij} + \Gamma^k_{j\sigma,i} Z^\sigma + \boxed{\Gamma^k_{j\sigma} Z^\sigma_{,i}} + \Gamma^k_{\tau i} Z^\tau_{,j} + \Gamma^k_{\tau i}\Gamma^\tau_{j\sigma} Z^\sigma$$

$$- \cancel{\Gamma^\tau_{ji} Z^k_{,\tau}} - \Gamma^\tau_{ji}\Gamma^k_{\tau\sigma} Z^\sigma$$

$$\left(\nabla_j \nabla_i Z\right)^k - \left(\nabla_i \nabla_j Z\right)^k = \left\{\Gamma^k_{\sigma j,i} - \Gamma^k_{\sigma i,j} + \Gamma^k_{\tau i}\Gamma^\tau_{\sigma j} - \Gamma^k_{\tau j}\Gamma^\tau_{\sigma i}\right\} Z^\sigma$$

$$\equiv \underbrace{R^k_{\sigma ij}}_{\binom{1}{3}\text{-tensor}} Z^\sigma$$

<u>Homework</u>: If $R_{\sigma i j}^{k} Z^{\sigma}$ transforms as a $\binom{1}{2}$-tensor $\forall$ vector $Z^{\sigma}\frac{\partial}{\partial x^{\sigma}}$ then $R_{\sigma i j}^{k}$ is a $\binom{1}{3}$-tensor