(11) ①

[11] $\boxed{\text{Symmetries of } R}$ (Assume Γ symmetric)

① $R^\alpha_{\beta j i} = - R^\alpha_{\beta i j}$

P.f. $R^\alpha_{\beta i j} Z^\beta = Z^\beta_{;i;j} - Z^\beta_{;j;i}$ ✓

② $R^\alpha_{[\beta i j]} = R^\alpha_{\beta i j} + R^\alpha_{j \beta i} + R^\alpha_{i j \beta}$

↑ cyclically permute

P.f $R^\alpha_{\beta i j} = \underbrace{\Gamma^\alpha_{\beta j, i}}_{\text{symmetric}} - \Gamma^\alpha_{\beta i, j} + \Gamma^\alpha_{\tau i} \underbrace{\Gamma^\tau_{\beta j}}_{\text{symmetric}} - \Gamma^\alpha_{\tau j} \Gamma^\tau_{\beta i}$

<u>Claim</u>: Any set of #'s $B_{\beta j i} = A_{\beta j i} - A_{\beta i j}$ (*)

symm

with $A_{\beta j i} = A_{j \beta i}$

satisfies $B_{[\beta j i]} = 0$ i.e. $B_{[\beta j i]} = \cancel{A_{\beta j i}} - \cancel{A_{\beta i j}} + \cancel{A_{i \beta j}} - \cancel{A_{i j \beta}} + \cancel{A_{j i \beta}} - \cancel{A_{j \beta i}}$

Since $A_{\beta j i} = \Gamma^\alpha_{\beta j, i}$ & $\bar{A}^\alpha_{\beta j i} = \Gamma^\alpha_{\tau i} \Gamma^\tau_{\beta j}$

are symm in βj & R^α ...

Let $R_{\alpha\beta ij} = g_{\alpha\sigma} R^{\sigma}{}_{\beta ij}$.

(3) $R_{\alpha\beta ij} = -R_{\beta\alpha ij}$

$= -R_{\alpha\beta ji}$ (from above)

i.e. antisymm in 1st 2 & last 2 indices

Proof: in Loc Lorentz ~~coords~~ coords,

$\left\{ \Gamma_{\alpha\beta j} = \frac{1}{2}\left\{ -g_{\beta j,\alpha} + g_{\alpha\beta,j} + g_{j\alpha,\beta} \right\} \right.$

$R_{\alpha\beta ij} = \Gamma_{\alpha\beta j,i} - \Gamma_{\alpha\beta i,j}$

$= \frac{1}{2}\left\{ -g_{\beta j,\alpha i} + \cancel{g_{\alpha\beta,ji}} + g_{j\alpha,\beta i} \right\}$

$- \left\{ -g_{\beta i,\alpha j} + \cancel{g_{\alpha\beta,ij}} + g_{i\alpha,\beta j} \right\}$.

$R_{\alpha\beta ij} = \frac{1}{2}\left\{ -g_{\beta j,\alpha i} + g_{\beta i,\alpha j} + g_{\alpha j,\beta i} - g_{\alpha i,\beta j} \right\}$ (*

from which (3) follows at once. $\{$get (3) & (4) from this$\}$

(Note: antisymm. is a coord indept prop of a

4 $R_{\alpha\beta ij} = R_{ij\alpha\beta}$ "Symm. under pr. exch."

Pf. This follows from (*). in Loc. Lor. fr.

But Symm under pair exch is a coord. indept property of a tensor. ✓ (FIP)

<u>Thm</u>: Symmetries 1 - 4 $\Rightarrow$ $\exists$ 20 indept entries in the Curvature tensor. (FIP)

<u>Project</u>: Show $\exists$ a metric with exactly 20 indept entries: [see David Meldgin]

- $R_{\alpha\beta ij}$ $\quad 4 \times 4 \times 4 \times 4 = 256$ entries
- $\left.\begin{array}{l} R_{\alpha\beta ij} = -R_{\beta\alpha ij} \\ R_{\alpha\beta ij} = -R_{\alpha\beta ji} \end{array}\right\}$ $\quad \underbrace{4 \times 4}_{6} \times \underbrace{4 \times 4}_{6}$ $\quad \leq 36$ indept entries

$$\begin{array}{cc} \begin{bmatrix} * & * & * & * & * & * \\ & * & * & * & * & * \\ & & * & * & * & * \\ & & & * & * & * \end{bmatrix} & \begin{array}{c} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ \hline 21 \end{array} \end{array}$$

- $R_{\alpha\beta ij} = R_{ij\alpha\beta}$ $\quad \Rightarrow \leq 21$ indept entries (FIP)
- $R_{\alpha[\beta ij]} = 0$ $\quad \Rightarrow$ 20 indept entries ... all others from

Ricci Tensor

$$R_{ij} = R^{\sigma}{}_{i\sigma j} \quad (\Rightarrow R_{ij} = R_{ji})$$

<u>Thm</u>: R_{ij} is the only non-trivial 2 tensor which can be obtained from the 4-tensor $R^{i}{}_{jkl}$ by contraction (modulo raising/lowering & trivial interchanges)

<u>Pf</u>: If $A_{ij} = -A_{ji}$, then

$$A^{i}{}_{j} = g^{\sigma i} A_{\sigma j} = -g^{\sigma i} A_{j\sigma} = -A_{j}{}^{i} \qquad (A)$$

$$A^{i}{}_{i} = \underbrace{g^{\sigma i}}_{\text{symm}} \underbrace{A_{\sigma i}}_{\text{antisym}} = 0 \qquad (B)$$

<u>Thus</u>: ① $R^{\sigma}{}_{\sigma ij} = R_{ij}{}^{\sigma}{}_{\sigma} = 0$ by (B)

② $R^{\sigma}{}_{i\sigma j} = -R_{i}{}^{\sigma}{}_{\sigma j} = R_{i}{}^{\sigma}{}_{j\sigma} = -R^{\sigma}{}_{ij\sigma}$

③ Since $R_{\alpha\beta ji} = R_{ji\alpha\beta}$ ② is equiv. to raising the latter two indices & contracting

<u>Note</u>: $R_{ij} = R_{ji} \Rightarrow R^{i}{}_{j} = R_{j}{}^{i} \equiv R^{i}_{j}$ $\left(\begin{matrix}\text{Not} \\ R^{i}{}_{j} = R^{j}{}_{i}\,!\end{matrix}\right)$

I.e., $R_{ij} = R^{\sigma}{}_{i\sigma j} = g^{\tau\sigma} R_{\tau i \sigma j} = g^{\tau\sigma} R_{\sigma j \tau i} = R_{ji}$

<u>Scalar</u> <u>Curvature</u>: $R = R^{\sigma}{}_{\sigma}$

<u>Note</u>: Since there's only "one way" to contract $R^{i}{}_{jkl}$ once, $\exists$ essentially only one way to contract it twice $\Rightarrow$ R is quite natural.