☒ Relativist & non-relativistic fluids

- Compressible Euler Equations ≈ Newton's Laws for a Continuous Media

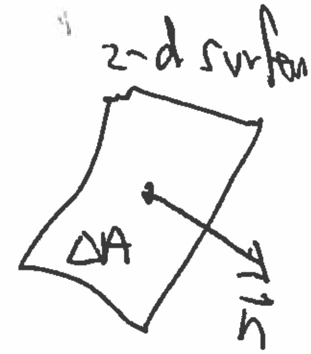
☒ Non-relativistic = classical version

(MA) $\rho_t + \operatorname{div} \rho\vec{v} = 0$ [cons of mass = continuity eqn]

(MO^i) $(\rho v^i)_t + \operatorname{div}[(\rho v^i)\vec{v} + p e^i] = 0$

Here: $\rho = \frac{\text{mass}}{\text{vol}}$, $\vec{v} = (v^1, v^2, v^3)$ = velocity

$p = \frac{\text{force}}{\text{area}}$ exerted by fluid



force ≈ $p \cdot \Delta A \, \vec{n}$

- Look to solve for $\rho(\vec{x},t)$, $\vec{v}(\vec{x},t)$ $\Big\{$ starting from
  - Unknowns: v^1, v^2, v^3, ρ, p — i-conds, ∂ conds, etc.

  Equations: 4

Need Eqn of state to close – barotropic $p = p(\rho)$

- The divergence thm tells us what they mean:
- <u>General case</u>: $q = \frac{\text{stuff}}{\text{vol}}$ is transported at velocity $\vec{v} = (v^1, v^2, v^3)$

<u>Defn</u>: $q\vec{v} \equiv$ "stuff flux vector"

<u>Claim</u>: $q_t + \operatorname{div}(q\vec{v}) = 0$ (*)

expresses "conservation of q"

I.e. Integrate (*) over 3-vol $\mathcal{V}$ at fixed t:

$$\iiint_{\mathcal{V}} q_t + \operatorname{div}(q\vec{v})\, \underbrace{d\mathcal{V}}_{dx^1 dx^2 dx^3} = 0$$

$$\frac{d}{dt}\iiint_{\mathcal{V}} q\, d\mathcal{V} + \iiint_{\mathcal{V}} \operatorname{div}(q\vec{v})\, d\mathcal{V} = 0$$

$$\Leftrightarrow \quad \underbrace{\frac{d}{dt}\iiint_{\mathcal{V}} \rho \, dV}_{\text{mass in } V} + \underbrace{\iint_{\mathcal{S}} \rho \underbrace{\vec{v}\cdot\vec{n}}_{\frac{\text{distance}}{\text{time}} \text{ normal to } \mathcal{S}} dS}_{\text{rate at which stuff passes outward thru Boundary of } \mathcal{V} \equiv \mathcal{S}} = 0$$

Conclude: $\rho_t + \operatorname{div} \rho\vec{v} = 0$ expresses that the time rate of change of stuff in any volume $\mathcal{V}$ = rate at which stuff passes outward thru $\partial\mathcal{V}$. $\Leftrightarrow$

Conservation of Stuff

- <u>Conclude</u>:

(MA) $\rho_t + \operatorname{div} \rho \underset{\sim}{\vec{V}} = 0 \equiv \frac{d}{dt} \iiint_{\mathcal{V}} \rho \, dV = -\iint_{\partial \mathcal{V} \equiv \mathcal{S}} \rho \vec{v} \cdot \vec{n} \, d$

$\equiv$ conservation of mass

$\equiv$ continuity equn

$$(\text{mo}^i) \quad (\rho v^i)_t + \operatorname{div} \underbrace{(\rho v^i \vec{V} + p \underset{\sim}{e}^i)} = 0$$

$\rho v^i \equiv \frac{\text{mass} \times \text{vel}}{\text{vol}}$ $\qquad (\rho v^i) \vec{V} \equiv$ i-mom flux vector

$\equiv$ i-mom density.

Use Div Thm to Interpret:

outward normal

$$\underbrace{\frac{d}{dt} \iiint_{\mathcal{V}} \rho v^i \, dv}_{\text{i-mom in vol } \mathcal{V}} = - \underbrace{\iint_{\mathcal{S}} \rho v^i u \cdot \vec{n} \, ds}_{\text{rate at which i-mom passes outward thru } \mathcal{S} = \text{boundary of } \mathcal{V}} - \underbrace{\iint_{\mathcal{S}} (p e^i \cdot \vec{n}) \, ds}_{\text{i-comp of the force on } \mathcal{S} \times (-1)}$$

- Conclude: (CE) $\Leftrightarrow$ Cons of Mass (MA)

& (MO) $\equiv$ changes in momentum are due only to the pressure forces

The equations:

$$\begin{pmatrix} \rho \\ \rho v^1 \\ \rho v^2 \\ \rho v^3 \end{pmatrix}_t + \operatorname{div} \begin{pmatrix} \rho v^1 & \rho v^2 & \rho v^3 \\ \rho v^1 v^1 + p & \rho v^1 v^2 & \rho v^1 v^3 \\ \rho v^2 v^1 & \rho v^2 v^2 + p, & \rho v^2 v^3 \\ \rho v^3 v^1 & \rho v^3 v^2 + & \rho v^3 v^3 + p \end{pmatrix}$$

or

$$\operatorname{Div}_{t,\vec{x}} \underbrace{\begin{pmatrix} \rho & \rho v^1 & \rho v^2 & \rho v^3 \\ \rho v^1 & \rho v^1 v^1 + p & \rho v^1 v^2 & \rho v^1 v^3 \\ \rho v^2 & \rho v^2 v^1 & \rho v^2 v^2 + p & \rho v^2 v^3 \\ \rho v^3 & \rho v^3 v^1 & \rho v^3 v^2 & \rho v^3 v^3 + p \end{pmatrix}^{\alpha\beta}}_{4\times 4 \text{ symmetric Tensor } T^{\alpha\beta} = T_{CE}^{\alpha\beta}}$$

Note: In frame where $v = 0$, $T = \begin{bmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{bmatrix} \ldots$

- For GR, Einstein wants eqn of form

$$G = \kappa T$$

Looks for $\mathrm{Div}\, G = 0$ so

$$\mathrm{Div}\, T = 0$$

is the relativistic version of compressible Euler

<u>Claim</u>: The relativistic version of T_{CE} is

$$T^{\alpha\beta} = (\rho + p)\, u^{\alpha} u^{\beta} + p\, g^{\alpha\beta} \qquad (*)$$

metric defines the gravitational field.

No Gravity = Flat space $\quad g^{\alpha\beta} = \eta^{\alpha\beta} \equiv \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$

$(*) \equiv$ the unique tensor $= \begin{bmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{bmatrix}$ in frame where $v = 0$.