Derivation of Einstein Curvature Tensor G

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Theorem: $G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R$ is the only nontrivial 2-tensor constructable from R^{σ}_{ijk}, 6 met using raisings & lowerings & contractions such that $\operatorname{div} G = 0$ as a consequence of the Bianci Identities.

Einstein Equation $G = \kappa T$ (E)

$T \equiv$ stress energy tensor $\begin{bmatrix} e & e\text{-flux} \\ \underset{\sim}{p} & \underset{\sim}{p}\text{-flux} \end{bmatrix}_{ij}$

$G \equiv$ Einstein Curvature tensor

$\operatorname{div} G = 0 \Rightarrow$ sol'n's of (E) satisfy

$\operatorname{div} T = 0 \approx\!> $ conservation (local) of energy and momentum.

We look for a 2-tensor G_{ij} constructable from R_{ij} and g_{ij} such that $(\operatorname{div} G)_j = G^{\sigma}{}_{j;\sigma} = 0$. We have

Bianchi: $$R^{i}{}_{j[kl;m]} = 0$$

$$\left\{ R^{i}{}_{jkl;m} + R^{i}{}_{jmk;l} + R^{i}{}_{jlm;k} \right\} = 0$$

contract i & m:

$$g^{jk}\left\{ R^{\sigma}{}_{jkl;\sigma} + R^{\sigma}{}_{j\sigma k;l} + R^{\sigma}{}_{jl\sigma;k} \right\} = 0$$

contract in $j\,k$:

(Check in loc int. fr)

The covariant deriv. of g is zero $\Rightarrow$

(B) $$\left(g^{jk} R^{\sigma}{}_{jkl}\right)_{;\sigma} + \left(g^{jk} R^{\sigma}{}_{j\sigma k}\right)_{;l} + \left(g^{jk} R^{\sigma}{}_{jl\sigma}\right)_{;k} = 0$$

$$g^{jk} R^{\sigma}{}_{jkl} = g^{jk} g^{\sigma\tau} R_{\tau jkl} = -g^{\sigma\tau} g^{jk} R_{j\tau kl}$$

$$= -g^{\sigma\tau} R_{\tau l} = -R^{\sigma}{}_{l}$$

$$g^{jk} R^{\sigma}{}_{j\sigma k} = g^{jk} R_{jk} = R$$

$$g^{jk} R^{\sigma}{}_{jl\sigma} = -g^{jk} R^{\sigma}{}_{j\sigma l} = -g^{jk} R_{jl} = -R^{k}_{l}$$

so (B) becomes:

$$-R^{\sigma}{}_{l;\sigma} + R_{;l} - R^{k}{}_{l;k} = 0$$

$$\Leftrightarrow \quad R^{\sigma}_{l\,;\sigma} - \frac{1}{2} R_{;l} = 0$$

$$\Leftrightarrow \quad R^{l\sigma}{}_{;\sigma} - \frac{1}{2}\left(g^{l\sigma} R\right)_{;\sigma} = 0$$

$$\Leftrightarrow \quad \underbrace{\left(R^{l\sigma} - \frac{1}{2} g^{l\sigma} R\right)}_{G^{l\sigma}}{}_{;\sigma} = 0$$

$G^{ij} = R^{ij} - \frac{1}{2} g^{ij} R$ satisfies $\operatorname{div} G = 0$

$$\boxed{G = R - \frac{1}{2} g \underset{\sim}{R}}$$

- In empty space: $T=0 \Rightarrow G=0$

<u>Claim</u>: $G_{ij}=0$ iff $R_{ij}=0$

<u>Pf</u>:

$$G_{ij} = R_{ij} - \tfrac{1}{2} R g_{ij}$$

$$G^{\sigma}_{\ \sigma} = R^{\sigma}_{\ \sigma} - \tfrac{1}{2} R \underbrace{g^{\sigma}_{\ \sigma}}_{id^{\sigma}_{\ \sigma}=4} = -R$$

$\therefore\quad G_{ij}=0 \Rightarrow G^{\sigma}_{\ \sigma}=0 \Rightarrow R^{\sigma}_{\ \sigma}=0=R \Rightarrow G_{ij}=R_{ij}=0$

$R_{ij}=0 \Rightarrow G_{ij}=0$ (easy) ✓ (Homework)

<u>Conclude</u>: $R_{ij}=0$ empty space field eqn's

Replaces: $\Delta \Phi = 0$ $\quad \Phi$ the gravitational potential

$$\nabla \Phi = -\frac{\text{force}}{\text{mass}}$$ in Newton Theory.