Special GR solutions

- Schwarzschild metric

(S) $$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

line ele on sphere

$r_s = 2GM$

$K = \frac{8\pi G}{c^4}$ $c=1$

- Much more complicated rotating black-hole soln of Kerr — (axially symmetric)

(K) $$ds^2 = -\left[1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right](du + a\sin^2\theta\, d\phi)^2$$

$$+ 2(du + a\sin^2\theta\, d\phi)(dr + a\sin^2\theta\, d\phi)$$

$$+ (r^2 + a^2\cos^2\theta)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

(Kerr – 1963)

No simple derivation. Other coord systems...

·bility of Kerr open problem < lots of research effort

- Reissner-Nordstrom metric – "non-rotating black hole w static electric & magnetic charge"

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2$$

$$\Delta = 1 - \frac{2GM}{r} + \frac{G}{r^2}(Q^2 + P^2)$$

$$\left.\begin{array}{l} Q = \text{electric charge of black hole} \\ P = \text{magnetic charge} \end{array}\right\} \text{by Gauss's Law}$$

- Friedmann Spacetime (dynamical)

$$\text{(F)} \quad ds^2 = -dt^2 + R(t)^2 \left\{ \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right\}$$

Thm: (Birkoff) The only empty space vacuum soln of $G_{ij} = 0 \Leftrightarrow R_{ij} = 0$ is Schwarzschild $\Rightarrow$ "no spherial gravity waves". Gravitational radiation is not present in simplest setting ~

Steps to Pf

General Spherically symmetric spacetime:

$$ds^2 = -B(r,t)\,dt^2 + \frac{1}{A(r,t)}\,dr^2 + E(r,t)\,dr\,dt + C(r,t)^2\,d\Omega^2$$

$\exists$ change of coordinate w. $\bar{r} = C(r,t)$

$$\Leftrightarrow \quad \boxed{ds^2 = -B(\bar{r},\bar{t})\,d\bar{t}^2 + \frac{d\bar{r}^2}{A(\bar{r},\bar{t})} + \bar{r}^2\,d\Omega^2}$$

(SSC)

Standard Schwarz Coords.

Plug (SSC) into $G = 8\pi T$, assume $T=0$, get (S).

Cor: Outside a spherically symmetric solution of $G = \kappa T$ representing any dynamical star, if $T=0$, its Schwarzschild

$\Rightarrow$ no spherical waves in empty space

~~Cosmology:~~

~~$ds^2 =$~~

- Static Fluid Sphere – (Oppenheimer–Volkov $\approx$ 1930's)

$$ds^2 = -B(r)\,dt^2 + \frac{1}{A(r)}\,dr^2 + r^2 d\Omega^2$$

"If $\frac{2GM(r)}{r} = 1$ Black Hole"

$A(r) = 1 - \frac{2GM(r)}{r}$

Given soln, radius r, $M(r)$ = mass inside radius r

$r_s = 2GM(r)$. If ever $r < \frac{9}{8} r_s \Rightarrow$ no static soln (Buchdahl Stab Limit)

⑤

<u>Cosmology</u>: (F) $r = \text{const}$ gives the position of galaxies in free fall.

<u>Check</u>: $r = \text{const}$ really is geodesic ($k=0$)

$$ds^2 = -dt^2 + R(t)^2 (dr^2 + r^2 d\Omega^2)$$

<u>Curve</u>: $t(\xi), r(\xi), \theta(\xi), \varphi(\xi)$.

$$g_{ij} = \begin{bmatrix} -1 & & & 0 \\ & R & & \\ & & r^2 R & \\ 0 & & & r^2 \sin\theta R \end{bmatrix}$$

Assume $r(s) = r_0$, $\dot{\theta} = \dot{\varphi} = 0$.

$$\underset{\sim}{x}(\xi) = (t(\xi), r_0, \varphi_0, \theta_0)$$

$$\frac{d\underset{\sim}{x}^i}{d\xi} = X^i = \left(\frac{dt}{d\xi}, 0, 0, 0\right)$$

$$\left\|\frac{d\underset{\sim}{x}}{ds}\right\| = \|X\| = -\left(\frac{dt}{d\xi}\right)^2$$

Assume $\xi = s \Rightarrow$ geodesic

~~Geode~~

Geodesic : $\frac{d^2x^i}{ds^2} = \Gamma^i_{jk} \dot{x}^i \dot{x}^j$

$x^0 = t$
$x^1 = r$
$x^2 = \phi$
$x^3 = \theta$

$\therefore$ only $\dot{x}^0 \neq 0$. $\therefore$ RHS zero except

$$\frac{d^2x^0}{ds^2} = \Gamma^0_{00} (\dot{t})^2$$

But : $\Gamma^0_{00} = \frac{1}{2} g^{0\sigma} \left\{ -g_{00,\sigma} + g_{\sigma 0,0} + g_{0\sigma,0} \right\}$

diag so
$1 = g^{00} \neq 0$
$g^{0\sigma} = 0 \quad \sigma \neq 0$

$= \frac{1}{2} \left\{ -\cancel{g_{00,0}} + \cancel{g_{00,0}} + g_{00,0} \right\}$

$= \frac{1}{2} g_{00,0} = 0$

because g_{00} indept of t. $\therefore$

geodesic if $\frac{d^2x^0}{ds^2} = 0 \Rightarrow x^0 = s, \; x^i = \text{const}$
is a geodesic ✓

Cosmology

- Hubbles Law: The galaxies are receding from us at a rate proportional to the distance:

Observational Fact:

$$V = H L \qquad (H)$$

velocity at which galaxy is receding

Hubbles "Constant"

distance from us to galaxy

$$H \approx h_0 \cdot \frac{100\,km}{s} \frac{1}{Mpc} \qquad h_0 \in [.5, 1]$$

Quoted value: $h_0 = .55$ (Niel Cornish)

$Mpc = 10^6 pc \qquad pc \cong 3.26$ ltay

"A galaxy 3.26 million ltys away is receding from us at about 55 km/s"

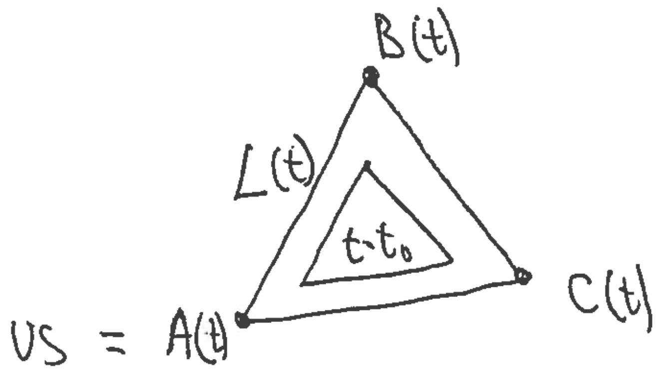
- Note: $[H] = \left[\frac{km}{s\ Mpc}\right] = \frac{1}{T}$

h_0 dimensionless constant

$$\frac{100\ km}{Mpc} \cong 3.24 \times 10^{-18} \quad \text{dimensionless}$$

- (H) is consistent with idea that the universe is uniformly expanding

Ex:



Assume all distances are increasing by a factor R(t)

$$L(t) = L_0 R(t)$$

R(t) = Cosmological scale factor

Then a galaxy at $B(t)$ will appear to recede from us at velocity

$$v = \frac{dL}{dt} = \frac{d}{dt}\left(R(t)L_0\right) = \dot{R}L_0 = \frac{\dot{R}}{R}\underbrace{L_0 R}_{L}$$

<u>Conclude</u>: uniform expansion implies (H) if we take

$$H_0 = \frac{\dot{R}}{R}$$

<u>Note</u> if $\frac{\dot{R}}{R} = H_0 \equiv$ constant, then

$$\ln(R/R_0) = H_0(t-t_0)$$

$$R = R_0\, e^{H_0(t-t_0)}.$$

(Steady state model of cosmology)

- It is believed that $H \equiv H(t)$ evolves with time.

- <u>Theorem</u>: The following metric defines a spacetime in which the 3-d space at $t=\text{const}$ uniformly expands according to the scale factor $R(t)$ & is symmetric homogeneous & uniform about each pt (FRW)

$$ds^2 = -dt^2 + R(t)^2 \left\{ \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right\}$$

Friedmann-Robertson-Walker metric.

<u>Note</u> ① The space at $t=\text{const}$ with metric $ds^2 = R(t)^2 \left\{ \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right\}$ defines a 3-d space of constant scalar curvature: $\text{Sign}(k) = \text{Sign}(R^{\sigma\tau}{}_{\sigma\tau})$

$R^{\sigma\tau}{}_{\sigma\tau}$ (See Weinberg)

Note ②: By rescaling the radial coordinate by (chng of coords')

$$\bar{r} = \alpha r, \quad \alpha = \text{const}$$

we can rescale k to value $k = -1, 0, 1$ (FLP).

"Pf of Theorem" In the case $k=0$, we obtain $ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$ from $dx^2+dy^2+dz^2$ by changing to Spherical coordinates:

$$z = r\cos\theta$$
$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$

Thus the effect of $R(t)$ is to rescale

all lengths by factor of $R(t)$ ✓.

For $k \neq 0$ see Weinberg.

- Assume the matter in universe has some average density $\rho(t)$ & pressure $p(t)$.

We look for metrics of form (FRW) that solve the Einstein equations

$$G = \frac{8\pi G}{c^4} T$$

where

$$T_{ij} = (\rho + p) u_i u_j + p g_{ij}$$

stress tensor for a perfect fluid.

- Assume (1) the galaxies are fixed relative to the spacial coordinates (r, θ, φ). This implies that the distance betw galaxies increases by factor $R(t)$:

$$L = L_0 R(t)$$

$L_0 \equiv$ distance apart when $R(t) = 1$.

(FIP) Show that this implies that galaxies follow geodesics of (FRW) metric $\Rightarrow$ they are in "freefall".

- Assume (1) $\Rightarrow$ $u^i = (1, 0, 0, 0)$ "no spatial motion".

- Our Equations are thus:

$$G_{ij}[g_{ij}] = \frac{8\pi G}{c^4}(\rho+p)u_i u_j + p g_{ij}$$

$$g_{ij} = \begin{bmatrix} -1 & & & 0 \\ & \frac{R(t)^2}{1-kr^2} & & \\ & & R(t)^2 r^2 & \\ 0 & & & R(t)^2 r^2 \sin^2\theta \end{bmatrix}$$

$$u_i = (-1, 0, 0, 0) \;, \; u^i = (1, 0, 0, 0)$$

Calculation (Wein Pg 472) $\Rightarrow$

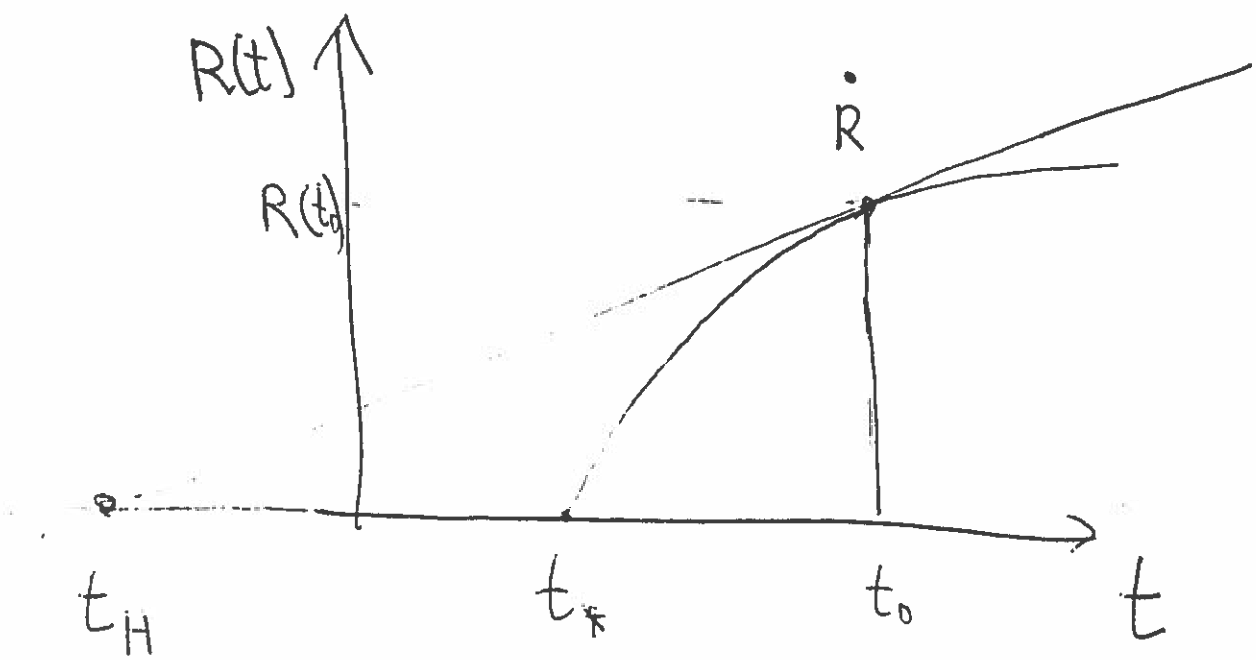
$$3\ddot{R} = -4\pi G(\rho+3p)R \qquad (1)$$

$$R\ddot{R} + 2\dot{R}^2 + 2k = 4\pi G(\rho-p)R^2 \qquad (2)$$

Eqn (1) $\Rightarrow$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) < 0$$

<u>Fact</u>: $R(t)$ is positive and concave down & increasing $\Rightarrow$ $R(t_*) = 0$ at some time in past $t_* \equiv$ "Big Bang"



$t_* > t_H$ $\qquad \frac{R(t_0)}{t_0 - t_H} = \dot{R}(t_0)$ $\qquad H_0 = \frac{\dot{R}(t_0)}{R(t_0)} = \frac{1}{t_0 - t_H}$

<u>Conclude</u>: The universe began at some time $t = t_*$, and

$$\text{Age of Universe} = |t_0 - t_*| < |t_0 - t_H| = \frac{1}{H_0}$$

$$H_0 = h_0 \cdot \frac{100\ \text{km}}{\text{s Mpc}} = h_0 \cdot 3.24 \times 10^{-18}\ \text{s}^{-1}$$

$$h_0 \approx .55$$

$$\frac{1}{H_0} = \frac{1}{h_0\, 3.24} \times 10^{18}\ \text{s}$$

$$1\ \text{year} = 3.15558 \times 10^{7}\ \text{s}$$

$$\frac{1}{H_0} = \frac{1}{(.55)(3.24)(3.15558)} \times 10^{11}\ \text{years}$$

$$= .1778 \times 10^{11}\ \text{years}$$

$$= 17.78 \times 10^{9}\ \text{years} \approx 18\ \text{billion years}$$