2 ①

General Relativity : $G = \kappa T$

Einstein curvature Tensor — $\kappa = \frac{8\pi G}{c^4}$ — Stress Energy Tensor

- $T \equiv$ energy density/momentum density & fluxes

$\rho = \frac{\text{energy}}{\text{vol}}$ $\rho u^i = \frac{\text{energy}}{\text{area time}}$ u^i = velocity

$T = T(\rho, u, p) = (\rho + p) u^i u^j + p g^{ij}$

- $G \equiv$ Einstein Curvature tensor

"Curvature of what?" Ans spacetime metric

Metric : $g = g_{ij} \, dx^i dx^j$ Tells how to measure distance in spacetime.

$g \equiv$ gravitational field The unknown in Einsteins Equations

$$\boxed{G[g] = \partial^2 g = T(\rho, u, p)}$$

Curvature of what? Spacetime metric g

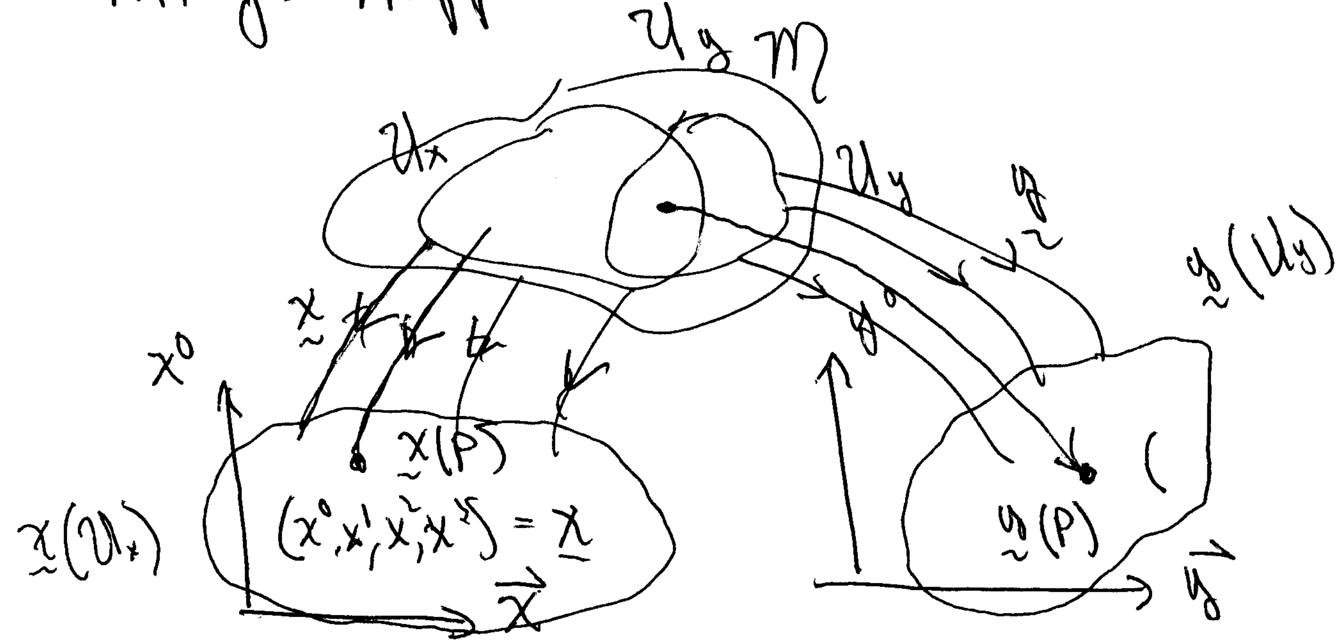
$\boxed{E}$ Gravitational metric

- Assumption: Spacetime is the manifold of events

Idea: events happen - like firecrackers going off at "positions in space" at a "given time"

$\mathcal{M} \equiv$ manifold of events

Assumption: you can place coordinate systems $(ct, x, y, z) \equiv (x^0, x^1, x^2, x^3)$ that name where & when things happen -



- $\mathcal{M}$ can be covered by coordinate charts

$$\underset{\sim}{x} : \mathcal{U}_x \xrightarrow{1-1} \mathbb{R}^4 \quad \text{(Called Atlas)}$$

- On the overlap of the charts

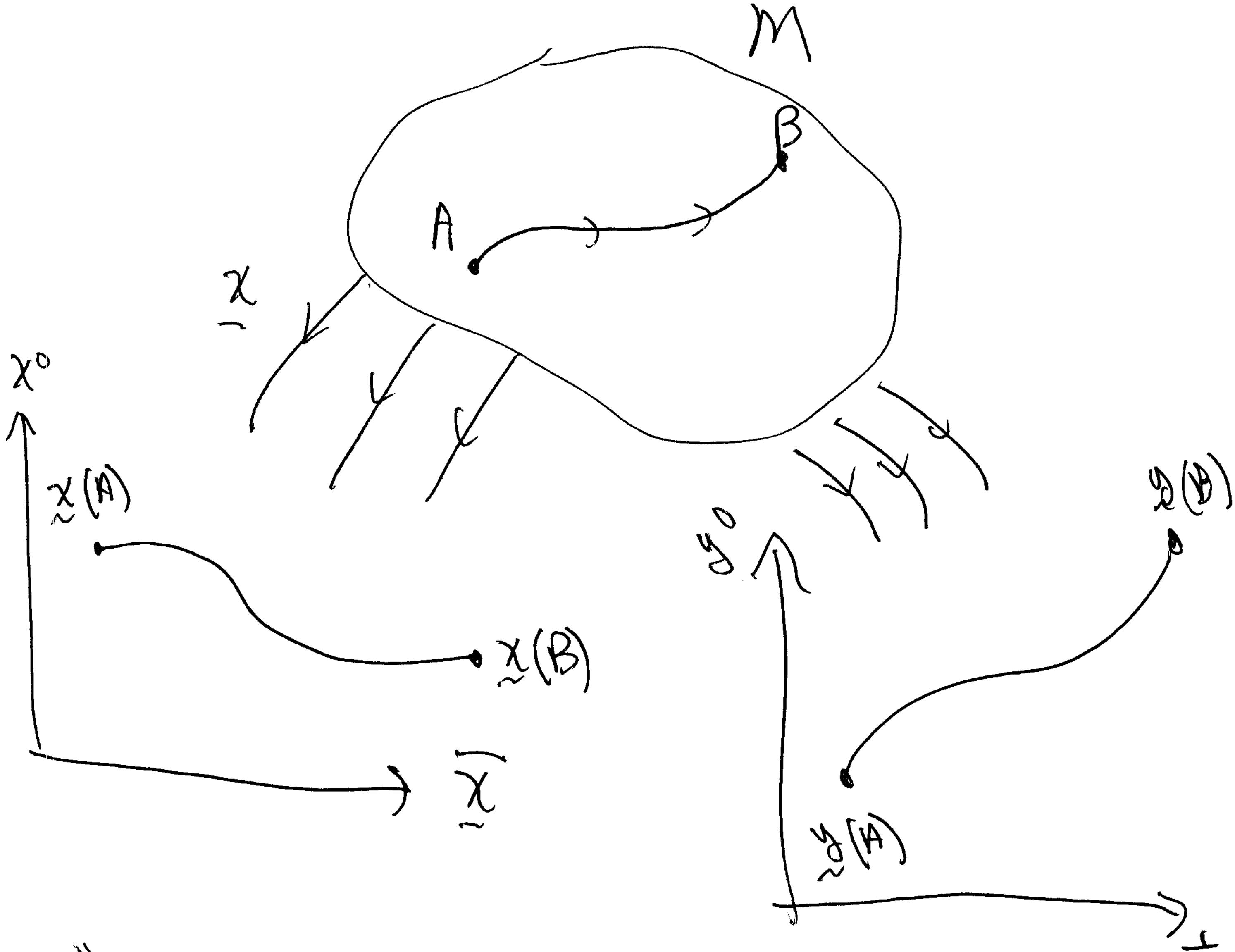
$$\underset{\sim}{y} \circ \underset{\sim}{x}^{-1} : \underset{\sim}{x}(\mathcal{U}_x) \xrightarrow[\text{onto}]{1-1} \underset{\sim}{y}(\mathcal{U}_y)$$

$$\underset{\sim}{x}(p) \longmapsto \underset{\sim}{y}(p)$$

We assume all the maps $\underset{\sim}{y} \circ \underset{\sim}{x}^{-1}$ betw coord charts are 1-1 onto invertible & infinity differentiable

- You can only study spacetime in a coordinate chart — In each chart you can make measurements etc, & we want the physical things we measure/calculate to be indept. of chart.

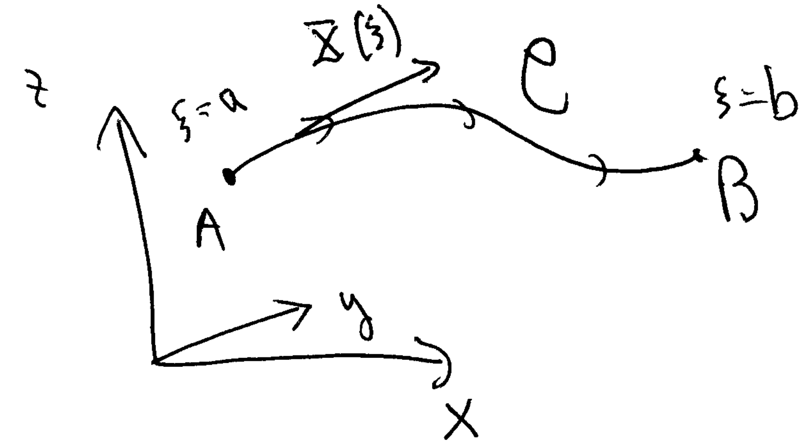
• For example: A rocket moves thru spacetime



The "length" of the path, or the time that ellapses on observers watch betw A & B should be the same in every coordinate system you ~~measure~~ calculate it in...

Q: How do you measure lengths in a coordinate system?

Ex: Consider Euclidean space $\mathbb{R}^3$

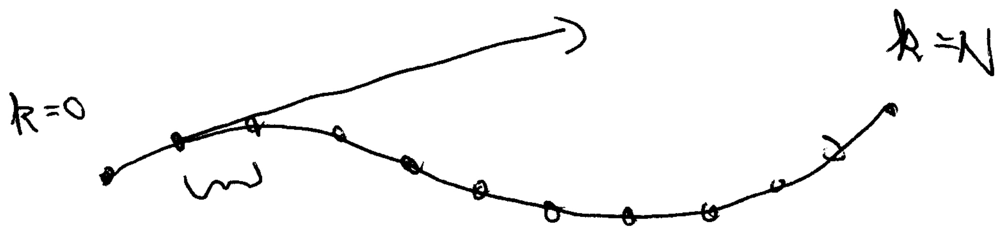


How do you calculate the length of curve?

- 1st parameterize curve –

$$\mathcal{C}: \ (x(\xi), y(\xi), z(\xi)) \equiv \vec{X}(\xi), \ a \leq \xi \leq b$$

- Tangent Vector: $\dfrac{d\vec{X}}{d\xi} = \overrightarrow{\left(\dfrac{dx}{d\xi}, \dfrac{dy}{d\xi}, \dfrac{dz}{d\xi} \right)} = \dot{\vec{X}}(\xi)$



$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2 + \Delta t^2} = \frac{\sqrt{\Delta x^2 + \Delta y^2 + \Delta t^2}}{\Delta \xi} \cdot \Delta \xi$$

$$= \sqrt{\left(\frac{\Delta x}{\Delta \xi}\right)^2 + \left(\frac{\Delta y}{\Delta \xi}\right)^2 + \left(\frac{\Delta t}{\Delta \xi}\right)^2}\ \Delta \xi$$

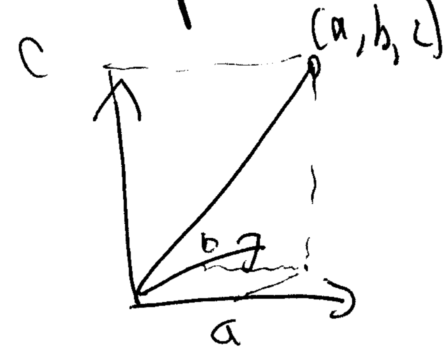
$$\text{Length} \approx \sum_{k=1}^{N} \Delta s_k = \sum_{k=1}^{N} \sqrt{\left(\frac{\Delta x_k}{\Delta \xi}\right)^2 + \left(\frac{\Delta y_k}{\Delta \xi}\right)^2 + \left(\frac{\Delta t_k}{\Delta \xi}\right)^2}\ \Delta \xi$$

$$L_A^B = \lim_{\Delta \xi \to 0} \sum_{k=1}^{N} \sqrt{(\ )^2 + (\ )^2 + (\ )^2}\ \Delta \xi = \lim_{\Delta \xi \to 0} \left[\begin{array}{c}\text{Riemann}\\ \text{sum}\end{array}\right]$$

$$= \int_a^b \left\| \frac{d\vec{X}}{d\xi} \right\| d\xi = \int_a^b \| X(\xi) \| \, d\xi$$

· Formula says – "The length is the integral of the length of the tangent vector wrt the parameter ξ": INDEPENDENT of Parameterization **

- How do you compute the length of a vector? You can use the dot product



$$X = \overrightarrow{(a, b, c)}$$

$$\|X\| = \sqrt{a^2 + b^2 + c^2}$$

$$\|X\|^2 = a^2 + b^2 + c^2$$

$$X \cdot X = \overrightarrow{(a, b, c)} \cdot \overrightarrow{(a, b, c)} = a^2 + b^2 + c^2 = \|X\|^2$$

$$= (a, b, c) \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\delta_{ij} \,=\, \text{identity matrix}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$X = \overrightarrow{(a, b, c)} = \overrightarrow{(a^1, a^2, a^3)}$$

$$Y = \overrightarrow{(b^1, b^2, b^3)}$$

$$\underline{X} \cdot \underline{Y} = (a^1, a^2, a^3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix}$$

$$= a^1 b^1 + a^2 b^2 + a^3 b^3$$

$$= a^i \delta_{ij} b^j$$ sum repeated up-down indiies

$\uparrow$ row $\uparrow$ colm

$$(a^1, a^2, a^3) \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix}$$

$$\begin{bmatrix} a^1(\delta_{11}, \delta_{12}, \delta_{13}) \\ + \\ a^2(\delta_{21}, \delta_{22}, \delta_{23}) \\ + \\ a^3(\delta_{31}, \delta_{32}, \delta_{33}) \end{bmatrix} \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix} = a^i \delta_{ij} b^j$$

HW Take 4x4 matrix A_{ij} & show $a^T A b = a^i A_{ij} b^j$

<u>Conclude</u>: $\|\underline{X}\|^2 = a^i \delta_{ij} a^j \quad a^i = \frac{dx^i}{d\xi}$

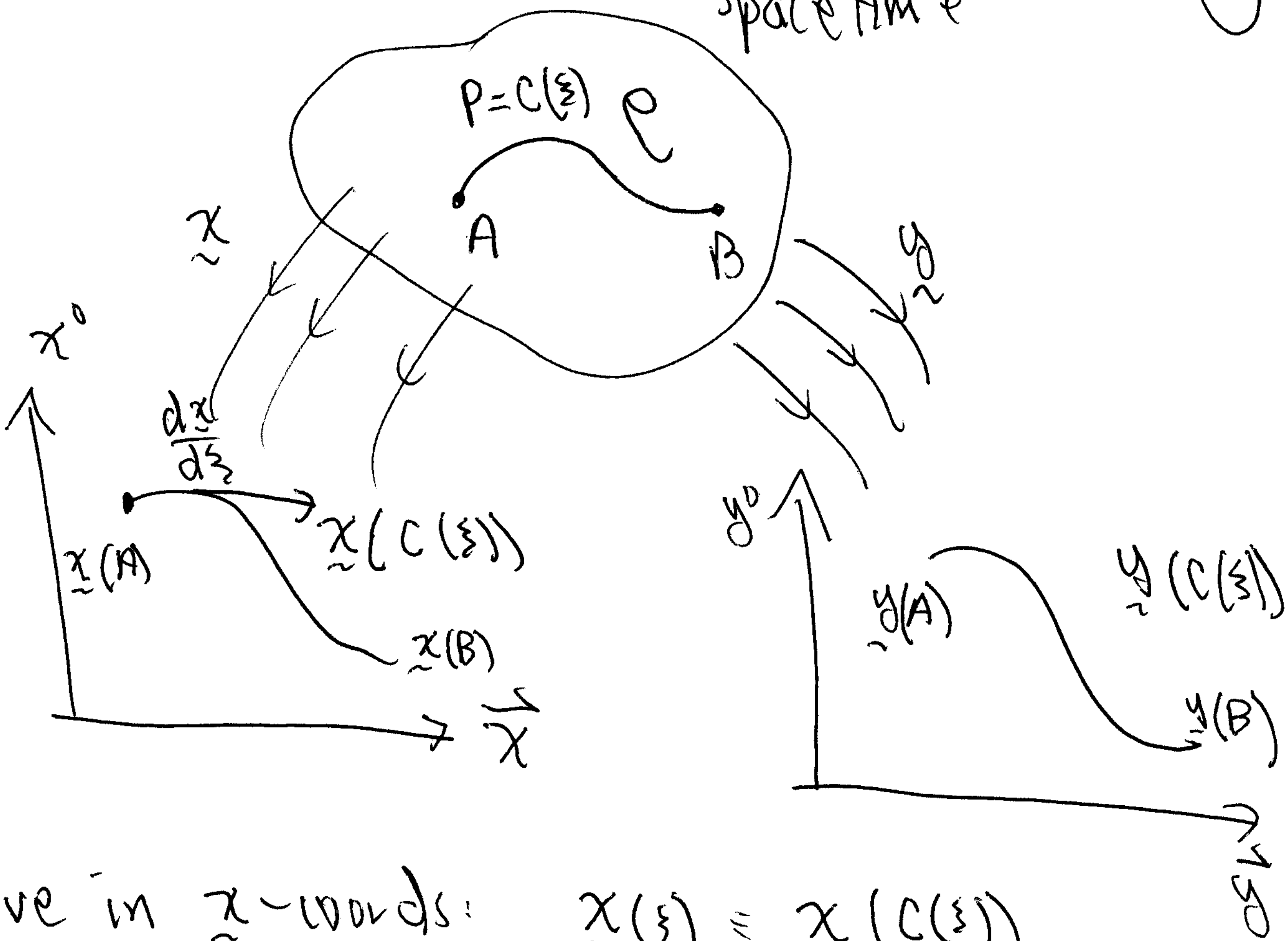
$$L_A^B = \int_a^b \|\underline{X}\| \, d\xi = \int_a^b \sqrt{\frac{dx^i}{d\xi} \delta_{ij} \frac{dx^j}{d\xi}} \, d\xi$$

<u>Conclude</u>: The dot product in $\mathbb{R}^3$ tells how to compute lengths in $\mathbb{R}^3$

- Q: What if the spacetim is curved, not Euclidean?

<u>Assumption</u> (GR) There exists in each coord system $\underset{\sim}{x} = (x^0, x^1, x^2, x^3)$ a symmetric 4×4 matrix field $g_{ij}(\underset{\sim}{x})$ such that

row colm



- Curve in $\underset{\sim}{x}$-coords: $\underset{\sim}{x}(\xi) \equiv \underset{\sim}{x}(C(\xi))$

$$\left(x^0(\xi), x^1(\xi), x^2(\xi), x^3(\xi)\right) = \underset{\sim}{x}(\xi)$$

$$\underline{X} = \frac{d\underset{\sim}{x}}{d\xi} = \text{tangent vector } (\dot{x}^0, \dot{x}^1, \dot{x}^2, \dot{x}^3)$$

$$\langle \underline{X}, \underline{X} \rangle = \dot{x}^i g_{ij} \dot{x}^j = \underset{\sim}{x}^T g \underset{\sim}{x} \qquad \dot{} \equiv \frac{d}{d\xi}$$

$$\quad 1\times 4 \quad 4\times 4 \quad 4\times 1$$

$$L_a^b = \int_a^b \|\underline{X}(\xi)\| \, d\xi \qquad \|\underline{X}\| = \sqrt{\langle X, X \rangle}$$

$$= \sqrt{\dot{x}^i g_{ij} \dot{x}^j}$$

This reduces to the Euclidean case when

$$g_{ij} = \delta_{ij} = \text{identity}.$$

We write $$\underbrace{\langle X, Y\rangle}_{\text{"inner product betw } X \,\&\, Y\text{"}} = \cancel{X}^i g_{ij} X^j$$

$$\begin{bmatrix} X^0 & \cdots & X^3 \end{bmatrix} \begin{bmatrix} g_{00} & \cdots & g_{03} \\ g_{10} & \cdots & g_{13} \\ g_{20} & \cdots & g_{23} \\ g_{30} & \cdots & g_{33} \end{bmatrix} \begin{bmatrix} X^0 \\ \vdots \\ X^3 \end{bmatrix}$$

Q: If g_{ij} is given in $\underset{\sim}{x}$-coords, what must g_{ij} be in $\underset{\sim}{y}$-coords in order that you compute <u>same</u> <u>length</u> in y coords?

<u>Ans</u>

$$g^y_{\alpha\beta} = \frac{\partial x^i}{\partial y^\alpha} g^x_{ij} \frac{\partial x^j}{\partial y^\alpha}$$

$$= \underset{4\times4}{J^T} \; \underset{4\times4}{g^x} \; \underset{4\times4}{J}$$

- A metric on spacetime is a 4x4 symmetric matrix $g_{ij}(x)$ defined at each $\underset{\sim}{x}$ in each coordinate system s.t.

$$g^{y}_{\alpha\beta}\Big|_{\underset{\sim}{y}(p)} = \frac{\partial x^i}{\partial y^\alpha}\, g^{x}_{ij}\, \frac{\partial x^j}{\partial y^\beta}\Big|_{\underset{\sim}{x}(p)}$$

Given this, $L_A^B = \int_{\xi_A}^{\xi_B} \|\dot{\gamma}(\xi)\|\, d\xi$ is the same number in every coordinate system, any parameterization of the curve

<u>Einstein Equation</u> $\quad G[g] = \frac{8\pi G}{c^4} T(\rho, u, p)$

The curvature of metric g

energy & the flow of energy

1

Q: How do you measure the curvature of g?

Ans Riemann Curvature Tensor

$$R^{i}_{\ jkl}$$

Q: What corresponds to ~~S~~ straight lines?

Ans Geodesics of g, curves that minimize the length!

Problem: Not every symmetric $g_{ij}(\underset{\sim}{x})$ can be a gravitational field. It has to be:

1. Invertible at every $\underset{\sim}{x}$
2. Lorentzian (Agree with special relativity in locally inertial coords)
3. Meet the Einstein Equations $G = \kappa T$

Q: How does g^{x}_{ij} transform to $g^{y}_{\alpha\beta}$?