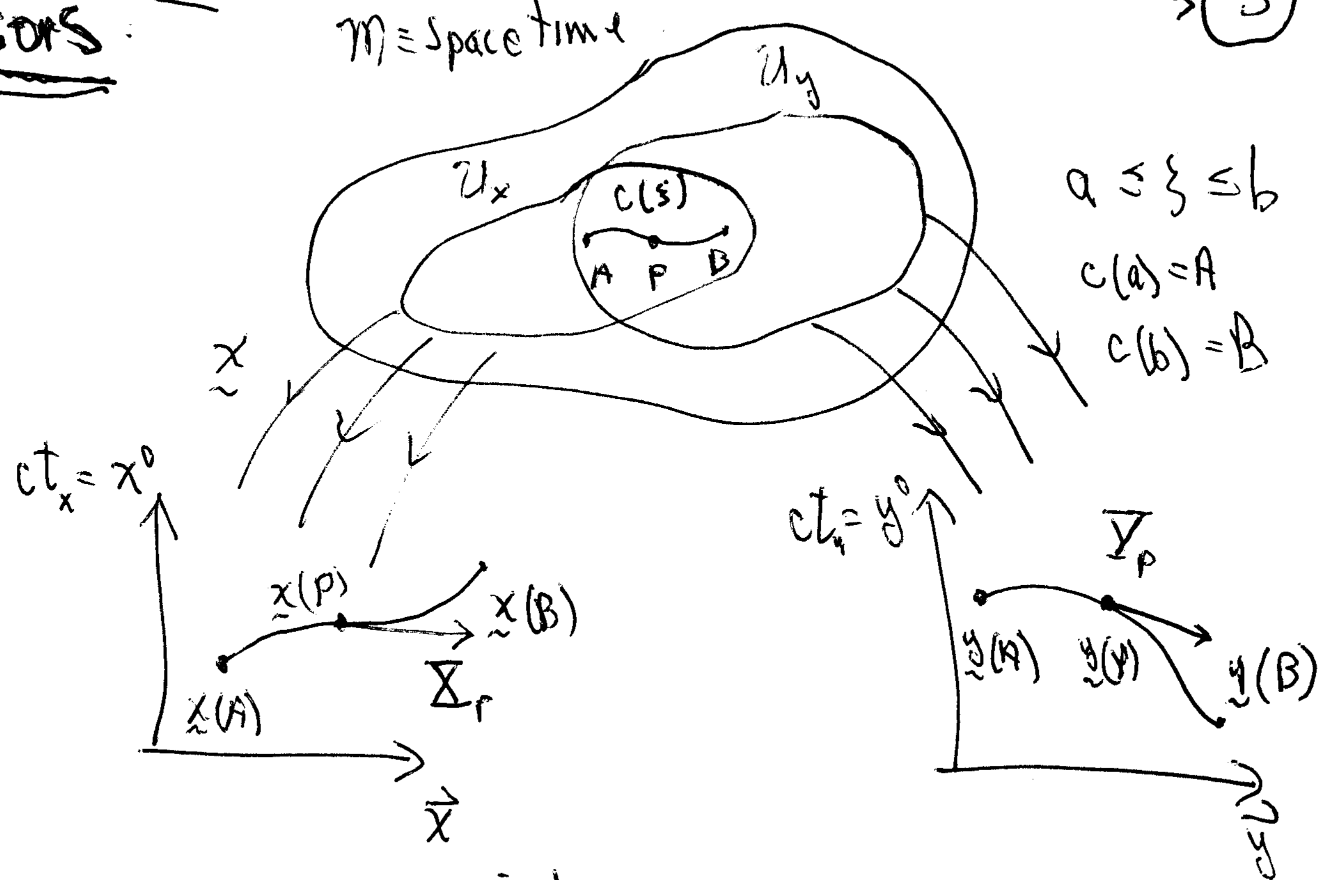
Vectors As Tensors: $\boxed{4}$

① §③



- The curve in $\underset{\sim}{x}$-coordinates:

$$\underset{\sim}{x}(c(\xi)) \equiv (x^0(\xi), x^1(\xi), \ldots, x^3(\xi)) = x^i(\xi) = \underset{\sim}{x}(\xi)$$

$i = 0,1,2,3$

The curve in $\underset{\sim}{y}$-coordinates:

$$\underset{\sim}{y}(c(\xi)) \equiv (y^0(\xi), \ldots, y^3(\xi)) = y^\alpha(\xi) = \underset{\sim}{y}(\xi) \qquad \alpha = 0,1,2,3$$

x-Tangent to curve: $\left.\dfrac{d}{d\xi} x^i(\xi)\right|_P = (\dot{x}^0(\xi), \ldots, \dot{x}^3(\xi)) = \dot{x}^i(\xi) = \dot{\underset{\sim}{x}}(\xi)$

($\mathbb{X}_P$ = $\underset{\sim}{x}$-coord tangent to curve @ $P = \dot{\underset{\sim}{x}}(\xi)\big|_P$)

y-Tangent to curve: $\left.\dfrac{d}{d\xi} y^\alpha(\xi)\right|_P = \dot{y}^\alpha(\xi) = \dot{\underset{\sim}{y}}(\xi)$

($\mathbb{Y}_P$ = $\underset{\sim}{y}$-coord tangent to curve @ $P = \underset{\sim}{y}(\xi)\big|_P$)

- Note: The curve $c(s)$ makes sense in space time.

$$c : \mathbb{R} \longrightarrow \mathcal{M}$$

since $\mathcal{M}$ is a set of points ("events" of spacetime)

But: The tangent to curve at P does not make sense in $\mathcal{M}$. It only makes sense as a vector in each coordinate system.

- Consider $\underline{X}_p \equiv$ x-tangent to $c(s)$ @ P.

$\underline{X}_p$ has a length and a direction

$$\underline{X}_p = a^0 \vec{e}_0 + a^1 \vec{e}_1 + a^2 \vec{e}_2 + a^3 \vec{e}_3$$

Here: (a^0, a^1, a^2, a^3) are the components of $\underline{X}_p$ wrt the coordinate basis $(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}^3)$.

Eg, $\vec{e}_0 = \overrightarrow{(1,0,0,0)}$, $\vec{e}_1 = \overrightarrow{(0,1,0,0)}$ etc

Note: The components are different from the coordinate unit vectors!

- In differential geometry we don't use the names $\vec{e}_i$ for coordinate unit vectors. We use

$$\frac{\partial}{\partial x^i} = \vec{e}_i = \overrightarrow{(0 \cdots 1 \cdots 0)}$$

$\uparrow$ ith component.

<u>Thus</u>: $\left\{ \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right\}_p$ is a basis for the tangent space of $\mathbb{R}^4$ @ $\underset{\sim}{x}(p)$.

<u>Note</u>: The <u>y</u>-coordinate basis $\left\{ \frac{\partial}{\partial y^\alpha} \right\}_{\alpha=0}^{3}$ @ p names a different set of directions in spacetime. Sometimes we write $\left.\frac{\partial}{\partial x^i}\right|_p$, $\left.\frac{\partial}{\partial y^\alpha}\right|_p$

Q How do the <u>components</u> of tangent vectors ~~to~~ on curves transform $x \to y$ @P & how do the <u>basic vectors</u> transform?

Ans As <u>tensors</u>, by the Jacobian $\left.\frac{\partial y^\alpha}{\partial x^i}\right|_p$

<u>Theorem</u>: The tangent vector to $c(s)$ @ p in x-coords is

$$\underline{X}_p = \dot{x}^i \frac{\partial}{\partial x^i}\Big|_p$$

and in y-coords is

$$\underline{Y}_p = \dot{y}^\alpha \frac{\partial}{\partial y^\alpha}\Big|_p$$

(Sum repeated up-down indices from 0,1,2,3)

This gives vector in terms of coord bases.

Moreover, $\dot{x}^i$, $\dot{y}^\alpha$, $\frac{\partial}{\partial x^i}$, $\frac{\partial}{\partial y^\alpha}$ transform by

$$\dot{y}^\alpha = \frac{\partial y^\alpha}{\partial x^i} \dot{x}^i \iff \underset{4\times 1}{\underset{\sim}{\dot{y}}} = \underset{4\times 4}{J} \cdot \underset{4\times 1}{\underset{\sim}{\dot{x}}} \qquad J = \frac{\partial y}{\partial x}$$

$$\frac{\partial}{\partial y^\alpha} = \frac{\partial x^i}{\partial y^\alpha}\frac{\partial}{\partial x^i} \iff \underset{1\times 4}{\frac{\partial}{\partial y}} = \underset{1\times 4}{\frac{\partial}{\partial x}} J^{-1} \qquad J^{-1} = \frac{\partial x}{\partial y}$$

<u>**Proof**</u>: To get $X_p = \dot{x}^i \frac{\partial}{\partial x^i}$ recall:

$$\underset{\sim}{x}(\xi) \equiv \underset{\sim}{x}(c(\xi)) \quad \text{curve in } x\text{-coords}$$

$$\underset{\sim}{x}(\xi) = (x^0(\xi), x^1(\xi), x^2(\xi), x^3(\xi))$$

$$\frac{d}{d\xi}\underset{\sim}{x}(\xi) \equiv \dot{\underset{\sim}{x}}(\xi) = (\dot{x}^0(\xi), \dots, \dot{x}^3(\xi))$$

$\therefore$ The vector $\overrightarrow{(\dot{x}^0(\xi), \dots, \dot{x}^3(\xi))}_p$ is tan. to $\mathcal{C}(\xi)$ at p

Conclude: in terms of the x-coord basis

$$X_p = \underbrace{\dot{x}^i(\xi)}_{\text{components}} \underbrace{\frac{\partial}{\partial x^i}}_{x\text{-coord. basis}} \qquad \text{sum } i = 0, 1, 2, 3$$

Similarly: $Y_p = \dot{y}^\alpha(\xi) \frac{\partial}{\partial y^\alpha}$

- To get the transformation laws: recall

$$\underset{\sim}{x}(\xi) = \underset{\sim}{x}(c(\xi)) \qquad x^i(\xi) = x^i(c(\xi))$$

$$\underset{\sim}{y}(\xi) = \underset{\sim}{y}(c(\xi)) \qquad y^\alpha(\xi) = y^\alpha(c(\xi))$$

so

$$\underset{\sim}{y}(c(\xi)) = \underbrace{\underset{\sim}{y} \circ \underset{\sim}{x}^{-1}}_{\substack{\underset{\sim}{y} \leftarrow \underset{\sim}{x} \\ \mathbb{R}^4 \leftarrow \mathbb{R}^4}} \circ \underbrace{\underset{\sim}{x} \circ c(\xi)}_{\underset{\sim}{x}(\xi)}$$

$$y^\alpha(\xi) = (\underset{\sim}{y}^\alpha \circ \underset{\sim}{x}^{-1}) \circ (\underset{\sim}{x} \circ c)(\xi)$$

$$\frac{d y^\alpha}{d\xi} = \dot{y}^\alpha = \frac{\partial y^\alpha}{\partial x^i} \frac{d x^i}{d\xi} = \underset{4\times 4}{J} \cdot \underset{4\times 1}{\dot{\underset{\sim}{x}}}$$

(Chain rule: $\frac{d}{d\xi} (\underset{\sim}{y}^\alpha \circ \underset{\sim}{x}^{-1}) \circ (\underset{\sim}{x} \circ c)(\xi)$

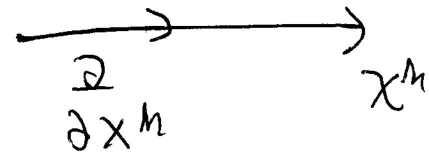
$$= \frac{\partial y^\alpha}{\partial x^i} \cdot \frac{d x^i}{d\xi} \qquad \text{sum } i = 0, 1, 2, 3$$

Eg $\frac{d}{d\xi} f(x^1(\xi), x^2(\xi)) = \frac{\partial f}{\partial x^1} \frac{dx^1}{d\xi} + \frac{\partial f}{\partial x^2} \frac{dx^2}{d\xi} = \frac{\partial f}{\partial x^i} \frac{dx^i}{d\xi}$)

To see how the bases transform:

Consider the x-coord unit vector $\frac{\partial}{\partial x^k}$ for some fixed $k = 0, 1, 2, 3$

This is the tangent vector to the coordinate axis, i.e. the curve



$$x^k(\xi) = \xi \qquad x^j(\xi) = 0 \quad j \neq k$$

$$\dot{x}^k(\xi) = 1, \quad \dot{x}^j(\xi) = 0 \quad j \neq k$$

Thus: $\frac{\partial}{\partial x^k} = X_p$ is tangent to curve $\underset{\sim}{x}(\xi)$ with $\dot{x}^k = 1$, $\dot{x}^j = 0$ $j \neq k$.

But we have for this curve, Y_p has comp's

$$\dot{y}^\alpha = \frac{\partial y^\alpha}{\partial x^i} \dot{x}^i = \frac{\partial y^\alpha}{\partial x^k} \dot{x}^k = \frac{\partial y^\alpha}{\partial x^k} \quad \left(\dot{x}^j = 0 \ \ j \neq k! \ \ \dot{x}^k = 1 \right)$$

$\therefore$ $X_p = \frac{\partial}{\partial x^k}$ means $Y_p = \dot{y}^\alpha \frac{\partial}{\partial y^\alpha} = \frac{\partial y^\alpha}{\partial x^k} \frac{\partial}{\partial y^\alpha}$

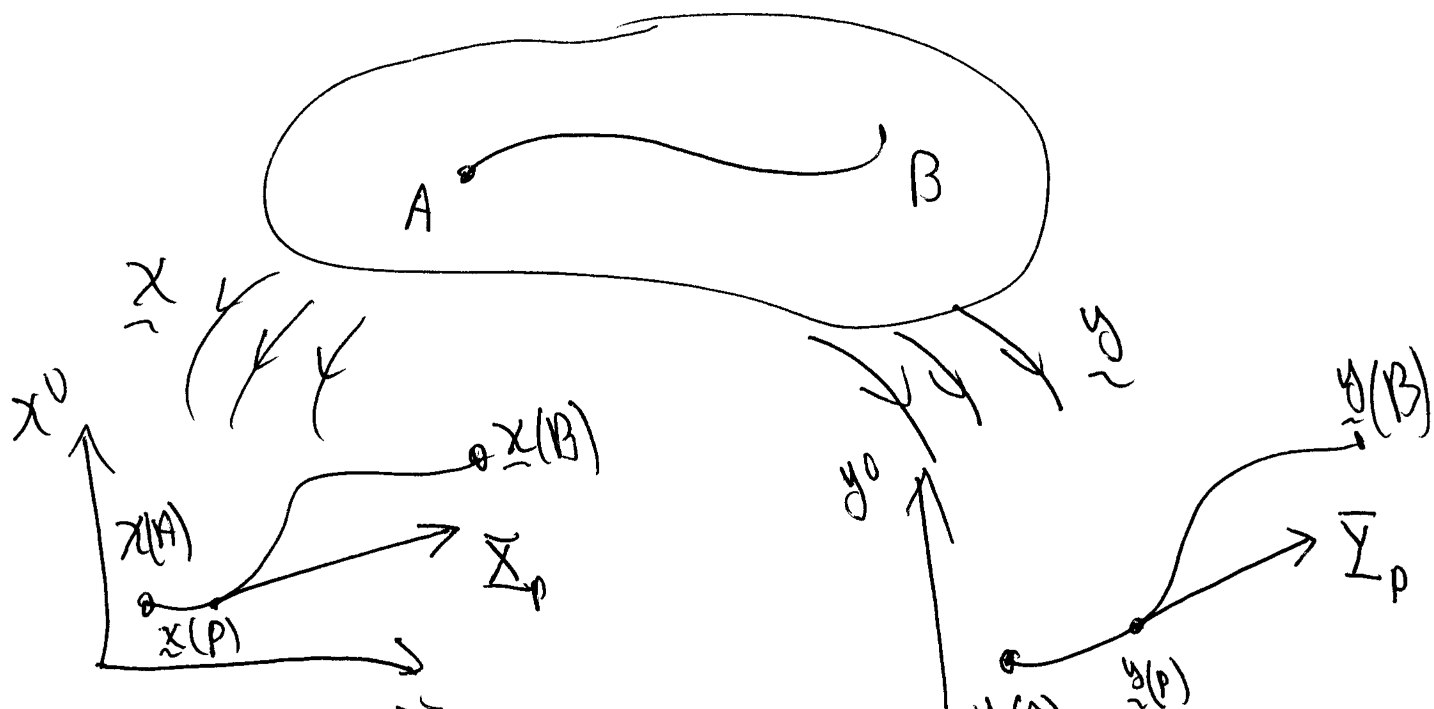
$X_p = Y_p$ means $\frac{\partial}{\partial x^k} = \frac{\partial y^\alpha}{\partial x^k} \frac{\partial}{\partial y^\alpha}$

• <u>Summary</u>: X_p & Y_p name the <u>same</u> vector at p iff

$$X_p = a^i \frac{\partial}{\partial x^i} \qquad Y_p = b^\alpha \frac{\partial}{\partial y^\alpha}$$

$$\left.\begin{aligned} &\text{where} \quad a^i = \frac{\partial x^i}{\partial y^\alpha} b^\alpha , \quad \frac{\partial}{\partial x^i} = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha} \\ &\Leftrightarrow \quad b^\alpha = \frac{\partial y^\alpha}{\partial x^i} a^i , \quad \frac{\partial}{\partial y^\alpha} = \frac{\partial x^i}{\partial y^\alpha} \frac{\partial}{\partial x^i} \end{aligned}\right\} (*)$$

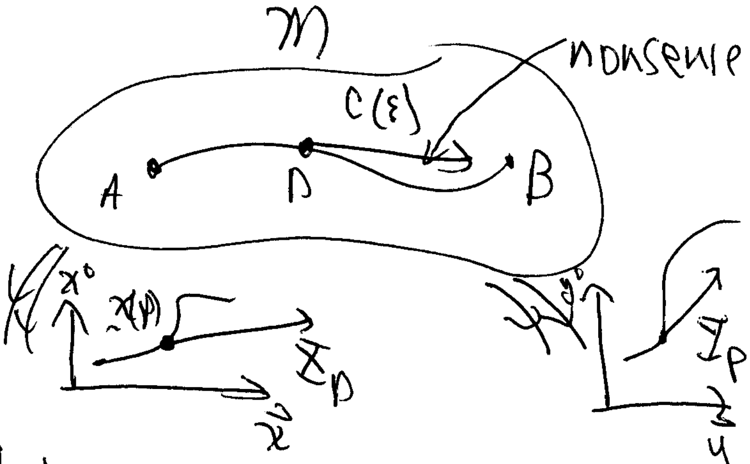
We say: $X_p = Y_p$ is the same vector iff (*).



$$\boxed{X_p = \dot{x}^i \frac{\partial}{\partial x^i} = \dot{y}^\alpha \frac{\partial}{\partial y^\alpha} = Y_p}$$

<u>Defn</u>: We say an up-index transforms <u>contravariantly</u> $\left(a^i = \frac{\partial x^i}{\partial y^\alpha} b^\alpha\right)$ and a down index transforms <u>covariantly</u> $\left(\frac{\partial}{\partial x^i} = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha}\right)$

Q How can we define $\underline{X}_p = \underline{Y}_p$ as somethin indept of coords? "$c'(\varepsilon)$" doesn't make sense



<u>Ans</u>: View

$a^i \frac{\partial}{\partial x^i} = b^\alpha \frac{\partial}{\partial y^\alpha}$ as

operator on functions;

$f: \mathcal{M} \to \mathbb{R}$ (makes sense)

$f \circ \underset{\sim}{x}^{-1} : \mathbb{R}^4 \to \mathbb{R}$ $\quad f \circ \underset{\sim}{y}^{-1} : \mathbb{R}^4 \to \mathbb{R}$ (makes sense)

$a^i \frac{\partial}{\partial x^i}(f \circ \underset{\sim}{x}^{-1}) = a^i \frac{\partial f}{\partial x^i} = b^\alpha \frac{\partial}{\partial y^\alpha}(f \circ \underset{\sim}{y}^{-1}) = b^\alpha \frac{\partial f}{\partial y^\alpha}$

"$a^i \frac{\partial}{\partial x^i} = b^\alpha \frac{\partial}{\partial y^\alpha}$" in the sense that they are the same differential operator on functions.