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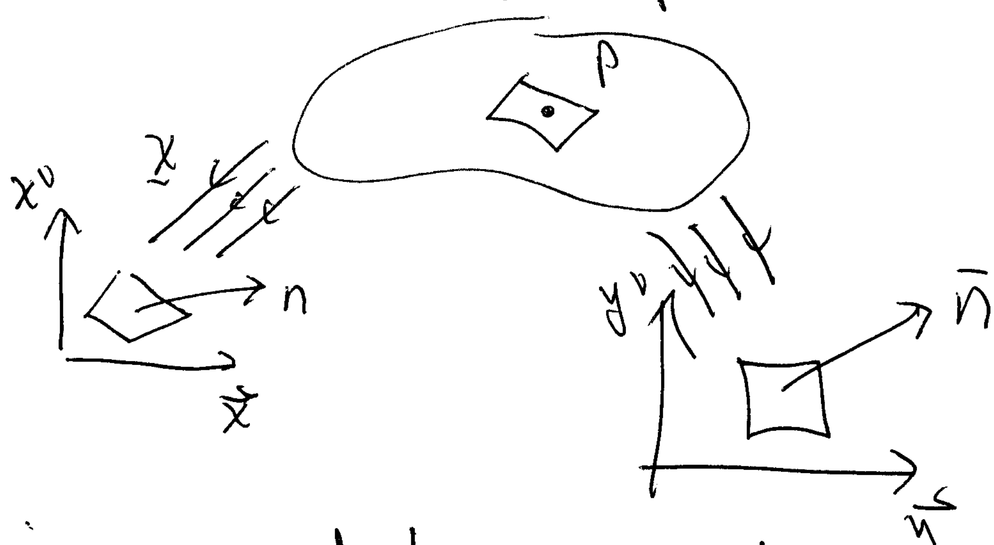
Covectors: Q: how do normals to hyperplanes transform?



- A hyperplane is spanned by 3 vectors

$Sp\{\bar{X}_1, \bar{X}_2, \bar{X}_3\}$ in x-coords $\Longleftrightarrow$ $Sp\{\bar{Y}_1, \bar{Y}_2, \bar{Y}_3\}$ in y-coords
$\mathcal{P}_x$ $\qquad\qquad\qquad\qquad$ $\mathcal{P}_y$

- In x-coords, let $n = (n_0, n_1, n_2, n_3)$ be the x-coord normal to hyperplane $\mathcal{P}$ at p

Specifically: $\bar{X}_1 = a_1^i \frac{\partial}{\partial x^i}$, $\bar{X}_2 = b_2^i \frac{\partial}{\partial x^i}$, $\bar{X}_3 = b_3^i \frac{\partial}{\partial x^i}$

Then $n \cdot \bar{X}_i = 0 \iff n_i a_k^i = 0 \quad k = 1, 2, 3$

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- Q: How do the normals to hyperplanes at P transform from x-coords to y-coords assuming that "$n \cdot \bar{X}_k = 0$" in x-coords $\Leftrightarrow$ "$\bar{n} \cdot \bar{Y}_k = 0$" in y-coords

Ans: $\bar{X}_k = a^i_k \frac{\partial}{\partial x^i} \overset{\text{iff}}{\Leftrightarrow} \bar{Y}_k = b^\alpha_k \frac{\partial}{\partial y^\alpha}$ same vect's

Want: $n_i a^i_k = 0 \quad k = 1,2,3$ iff $\bar{n}_\alpha b^\alpha_k = 0$

But to name same vectors we have

$$b^\alpha_k = \frac{\partial y^\alpha}{\partial x^i} a^i_k$$

Therefore we need

$$n_i a^i_k = 0 \text{ iff } \underbrace{\bar{n}_\alpha \frac{\partial y^\alpha}{\partial x^i}}_{n_i \text{ suffices}} a^i_k = 0$$

It suffices to define ~~Need~~: $\boxed{n_i = \bar{n}_\alpha \frac{\partial y^\alpha}{\partial x^i}} \Leftrightarrow \boxed{\frac{\partial x^i}{\partial y^\alpha} n_i = \bar{n}_\alpha}$

- More accurately, n keeps track of how the coordinate dot product (which tells angles & lengths in ~~the~~ each coordinate system) transforms betw coordinates.

That is: $n_i a^i_n = 0$ iff $\bar{n}_\alpha \bar{a}^\alpha = 0$

doesn't tell how the length of n transforms.

More accurately: ask that n transform so that

$$n_i a^i = \bar{n}_\alpha \bar{a}^\alpha$$

"$\vec{n} \cdot \vec{a}$" in x-coords "$\vec{\bar{n}} \cdot \vec{\bar{a}}$" in y-coords

$$= \underbrace{\bar{n}_\alpha \frac{\partial y^\alpha}{\partial x^i}}_{n_i} a^i \Rightarrow \boxed{\begin{aligned} n_i &= \frac{\partial y^\alpha}{\partial x^i} \bar{n}_\alpha \\ \bar{n}_\alpha &= \frac{\partial x^i}{\partial y^\alpha} n_i \end{aligned}}$$

Conclude: "coord normals to hyperplanes transform like co-vectors, not vectors"

- Of course – n_i transform differently at each point P because $\frac{\partial y^\alpha}{\partial x^i}$ depends on P.

- For vectors, $X_p = b^i \frac{\partial}{\partial x^i}\Big|_p$

  components ↗ ↖ coord bases

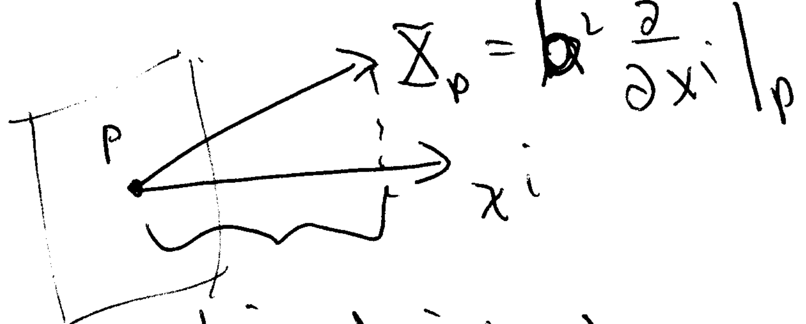
For co-vectors: $\omega_p = a_i \, \underline{dx^i}\Big|_p$

  components ↗ ↖ coord basis of 1-forms

Q: ~~How~~ How do we define $dx^i\Big|_p$?

... and how do they transform?

- Think of dx^i, $i = 0,1,2,3$ as measuring the change in the x^i coordinate <u>tangentially</u> at p. That is — the change in x^i along $\bar{X}_p$ is just b^i



$$b^i = dx^i(X_p)$$

<u>Define</u>: $\omega_p = a_i \, dx^i \big|_p$

"Computes the tangential change in x-coords in direction normal to $(a_0, a_1, a_2, a_3) = n$"

So: $\omega_p(\bar{X}_p) = a_i \, dx^i(\bar{X}_p) = \underbrace{a_i b^i}_{\text{"dot product in } \underline{x}\text{-coords"}}$

$= a_0 b^0 + \cdots + a_3 b^3$

Q: If $dx^i|_p$ is viewed as operating on vectors at p to compute the i-th component, $dx^i(X_p) = b^i$, then how do the dx^i transform?

$$n_i dx^i|_p \quad x\text{-coords} \overset{?}{\Longleftrightarrow} \bar{n}_\alpha dy^\alpha|_p \quad y\text{-coords}$$

$$X_p = b^i \frac{\partial}{\partial x^i}\Big|_p \Longleftrightarrow Y_p = \bar{b}^\alpha \frac{\partial}{\partial y^\alpha}\Big|_p$$

Want $n_i dx^i(X_p) = \bar{n}_\alpha dy^\alpha(Y_p)$

But: $dx^i(X_p) = b^i = \frac{\partial x^i}{\partial y_\alpha}\bar{b}^\alpha = \underbrace{\frac{\partial x^i}{\partial y_\alpha} dy^\alpha(Y_p)}_{dx^i}$

Condition: $\boxed{dx^i = \frac{\partial x^i}{\partial y_\alpha} dy^\alpha}$ $\boxed{\frac{\partial y^\alpha}{\partial x^i} dx^i = dy^\alpha}$

Conclude: At each $P \in M$, & coord systems x & y defined in a nbhd of P, we identify vectors by –

$$a^i \frac{\partial}{\partial x^i}\Big|_p = a^\alpha \frac{\partial}{\partial y^\alpha}\Big|_p$$

and covectors by

$$a_i dx^i \Big|_p = a_\alpha dy^\alpha \Big|_p$$

Though expressed in a coord system, we view them as naming direction in M @ P that are indept of coordinates.

Up indices transform contravariantly

Down indices transform co-variantly

- We say: $$T_pM = \mathrm{Span}\{\underbrace{\frac{\partial}{\partial x^0}, \dots, \frac{\partial}{\partial x^3}}_{\text{x-coord basis of } T_pM}\}\Big|_p$$

$$= \mathrm{Span}\{\underbrace{\frac{\partial}{\partial y^0}, \dots, \frac{\partial}{\partial y^3}}_{\text{y-coord basis of } T_pM}\}\Big|_p$$

T_pM = Tangent space of M at p

We say: $$T^*_pM = \mathrm{Span}\{\underbrace{dx^0, \dots, dx^3}_{\text{x-coord basis of } T^*_pM}\}\Big|_p$$

$$= \mathrm{Span}\{\underbrace{dy^0, \dots, dy^3}_{\text{y-coord basis of } T^*_pM}\}\Big|_p$$

T^*_pM Co-Tangent space of M at p.

$$X_p = b^i \frac{\partial}{\partial x^i}\Big|_p = \underline{\text{vector}}$$

$$\omega_p = a_i dx^i\Big|_p = \underline{\text{co-vector}}$$

We view covectors as acting linearly on vectors by

$$\omega_p(\underline{X}_p) = a_i\, dx^i\left(b^j \frac{\partial}{\partial x^j}\right)\Big|_p$$

$$= a_i b^i = \underset{\sim}{a} \cdot \underset{\sim}{b}$$

↑ dot product in $\underset{\sim}{x}$-coords

or $$\omega_p(\underline{X}_p) = \bar{a}_\alpha\, dy^\alpha\left(\bar{b}^\beta \frac{\partial}{\partial y^\beta}\right)$$

$$= \bar{a}_\alpha \bar{b}^\alpha = \bar{\underset{\sim}{a}} \cdot \bar{\underset{\sim}{b}}$$

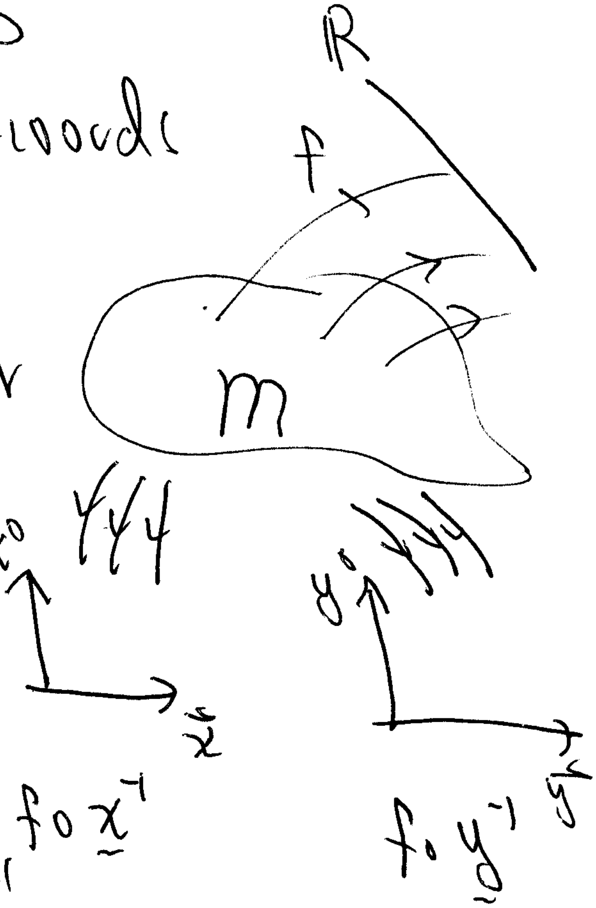
↑ dot product in $\underset{\sim}{y}$-coords

$\omega_p(\underline{X}_p) =$ "The tangential change in $\underline{X}_p$ moving normal to hyperplane ω_p"

Most Importantly:

~~Note~~: Covectors & vectors together keep track of how the coordinate dot product transforms $x \to y$

$$\underbrace{a_i b^i}_{\text{"}a \cdot b\text{" in } x\text{-coords}} = \underbrace{\bar{a}_\alpha \bar{b}^\alpha}_{\text{"}\bar{a} \cdot \bar{b}\text{" in } y\text{-coords}}$$



__Ex__: Let f be a scalar function defined on M

$$f : M \to \mathbb{R}$$

The x- & y-coord representations of f:

$$f \circ \underset{\sim}{x}^{-1} : \mathbb{R}^4 \to \mathbb{R} \equiv f(\underset{\sim}{x})$$

$$f \circ \underset{\sim}{y}^{-1} : \mathbb{R}^4 \to \mathbb{R} \equiv f(\underset{\sim}{y})$$

Consider now:

$$df = \frac{\partial f}{\partial x^i} dx^i \leftarrow$$ "The differential of a fn defined in $\mathbb{R}^n$ MATH 21D"

Q: is $df = \left.\frac{\partial f}{\partial x^i}\right|_p dx^i$ a 1-form in the co-tangent space of M at P?

Ans: Yes I.e.

$$df(\bar{X}_p) = \frac{\partial f}{\partial x^i} dx^i(\bar{X}_p) = \frac{\partial f}{\partial x^i} b^i$$

$$df(\bar{X}_p) = b^i \left.\frac{\partial f}{\partial x^i}\right|_p =$$ Directional derivative of f in direction $\bar{X}_p$ in x-coords —

$$df(\bar{Y}_p) = \frac{\partial f}{\partial y^\alpha} dy^\alpha(\bar{Y}_p) = \frac{\partial f}{\partial y^\alpha} \bar{b}^\alpha$$

$$= \frac{\partial y^\alpha}{\partial x^i} b^i \frac{\partial}{\partial y^\alpha} = b^i \frac{\partial}{\partial x^i} \checkmark$$

I.e., how does $\frac{\partial f}{\partial x^i}$ transform?

$$f(\underset{\sim}{x}) = f(x^0, \ldots, x^3)$$

$$\frac{\partial f}{\partial x^i} = \frac{\partial f}{\partial x^i}(x^0, \ldots, x^3)$$

If $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{y})$, then

$$\frac{\partial f}{\partial y^\alpha} = \frac{\partial}{\partial y^\alpha} f(x^0(\underset{\sim}{y}), \ldots, x^3(\underset{\sim}{y}))$$

$$= \frac{\partial f}{\partial x^0}\frac{\partial x^0}{\partial y^\alpha} + \cdots + \frac{\partial f}{\partial x^3}\frac{\partial x^3}{\partial y^\alpha}$$

$$= \frac{\partial x^i}{\partial y^\alpha}\frac{\partial f}{\partial x^i}$$

"Components of "gradiant f" should be viewed as components of 1-form" $\Rightarrow$

$$df = \frac{\partial f}{\partial x^i} dx^i$$

is covariant, $\nabla f = \overrightarrow{\left(\frac{\partial f}{\partial x^0}, \ldots, \frac{\partial f}{\partial x^3}\right)}$ is <u>not</u> (as a <u>vector</u>)

<u>Conclude</u>: $df(X_p)$ can be viewed as the 1-form df acting on tangent vector X_p, <u>or</u> equivalently as

$$df(X_p) = \frac{\partial f}{\partial x^i} a^i \Big|_p = a^i \frac{\partial}{\partial x^i}\Big|_p (f)$$

the gradiant of f in direction X_p

<u>Therefore</u>: To be ~~covari~~ defined indept of coordinate, $df = \frac{\partial f}{\partial x^i} dx^i$ needs to be viewed as a 1-form not $\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x^0}, \ldots, \frac{\partial f}{\partial x^3}\right)$ as ∇f !

Big motivation for introducing co-vectors!

☒ Riesz Representation Thm:

Thm: The bounded linear functionals on $\mathbb{R}^n$ can be represented by the inner product, and have the same dimension (as a vector space) as $\mathbb{R}^n$

Ex: $\vec{a} = \overrightarrow{(a^1, \ldots, a^n)} \in \mathbb{R}^n$ (vector space)

Let ω be a linear functional: $\omega \in \mathbb{R}^{n*}$ (dual space of $\mathbb{R}^n$)

$$\omega : \mathbb{R}^n \to \mathbb{R}$$

$$\omega(c_1\vec{a} + c_2\vec{b}) = c_1\omega(\vec{a}) + c_2\omega(\vec{b})$$

$$c_i \in \mathbb{R}, \quad \vec{a}, \vec{b} \in \mathbb{R}^n$$

then $\exists \ \vec{c} \in \mathbb{R}^n$ st $\omega(\vec{a}) = \vec{c} \cdot \vec{a}$

Since $\omega_c(\vec{a}) = \vec{c} \cdot \vec{a}$ is a linear fn'al $\forall \ \vec{c} \in \mathbb{R}^n$

This implies $(\mathbb{R}^n)^* \equiv \mathbb{R}^n$ has dim n.

Cor: T_p^*M is a vector space of dim n.

Pf. $\omega \in T_p^*M$ means $\omega = a_i \, dx^i|_p$ for some a_i (in coord system x).

But $\omega(c_1 X_1 + c_2 X_2) = c_1 \omega(X_1) + c_2 \omega(X_2)$

$$\left(\omega\left(b^i \frac{\partial}{\partial x^i}\right) = a_i b^i\right)$$

$\therefore$ $T_p^*(M)$ is the dual space of T_pM

$\Rightarrow$ both have dim $= n$ (as vector spaces)