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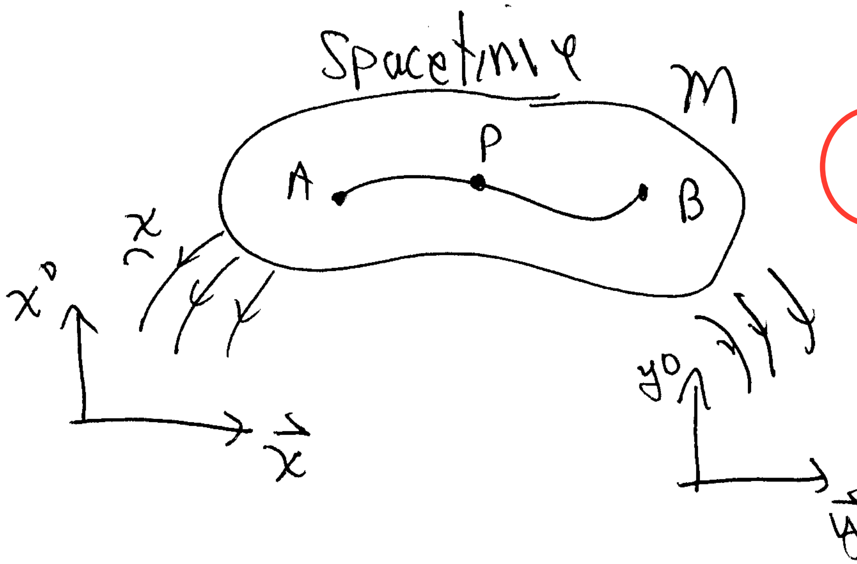
Metric:



Newton: $\cancel{M_E}\,\vec{a} = -G\dfrac{\cancel{M_E}M_S}{r^2}\dfrac{\vec{r}}{r} \Rightarrow$ the sun determines the paths

Einstein: The gravitational force creates the paths as geodesics of spacetime metric

$$\int_{\underset{\sim}{x}(A)}^{\underset{\sim}{x}(B)} \|\dot{\underset{\sim}{x}}(\xi)\|\, d\xi = L_A^B$$

Euclidean: $\|\dot{\underset{\sim}{x}}(\xi)\| = \sqrt{\sum_{i=0}^{3} |\dot{x}^i(\xi)|^2}$

$$= \sqrt{\dot{x}^i\,\delta_{ij}\,\dot{x}^j}$$

depends on the coordinates

Einstein/Riemann $\quad \| \dot{x}^i(\xi) \| = \sqrt{\dot{x}^i g_{ij} \dot{x}^j}$

$g_{ij}(\underset{\sim}{x})$ are the components of the metric in $\underset{\sim}{x}$-coords

Want: $\displaystyle\int_{\underset{\sim}{x}(A)}^{\underset{\sim}{x}(B)} \| \dot{\underset{\sim}{x}}(\xi) \| \, d\xi = L_A^B = \int_{\underset{\sim}{y}(A)}^{\underset{\sim}{y}(B)} \| \dot{\underset{\sim}{y}}(\xi) \| \, d\xi$

$\sqrt{\dot{x}^i g_{ij} \dot{x}^j} \qquad \sqrt{\dot{\underset{\sim}{y}}^\alpha \bar{g}_{\alpha\beta} \dot{y}^\beta}$

How should the g_{ij} transform so you get the same length computed in every coord system?

Ans: $\displaystyle \bar{g}_{\alpha\beta} = \frac{\partial x^i}{\partial y^\alpha} g_{ij} \frac{\partial x^j}{\partial y^\beta}$

$$\underset{4\times4}{\bar{g}} = \underset{4\times4}{J^T} \; \underset{4\times4}{g} \; \underset{4\times4}{J}$$

- More generally: $g_{ij}|_p$ computes an inner product between vectors in T_pM that generalizes the coord dot product —

<u>Defn</u>: $\langle X_p , Y_p \rangle = a^i g_{ij} b^j \big|_p$

$X_p = a^i \frac{\partial}{\partial x^i}\Big|_p \qquad Y_p = b^j \frac{\partial}{\partial x^j}\Big|_p$

- Defined at each p $\left(\begin{array}{l} g_{ij} \text{ depends on } \underset{\sim}{x}(p) \\ g_{ij} \equiv g_{ij}(\underset{\sim}{x}) \end{array} \right)$

~~• $\langle \, , \rangle$ is bilinear~~

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~~$\langle c_1 X_p + c_2 Y_p , Z_p \rangle$~~

- We always assume:

<u>symmetric</u>: $g_{ij} = g_{ji}$

so $\langle X_p, Y_p \rangle = a^i g_{ij} b^j = b^i g_{ij} a^j$

$= \langle Y_p, X_p \rangle$

<u>nondegenerate</u>: g_{ij} has <u>inverse</u> g^{ij}

so $g_{i\sigma} g^{\sigma j} = \delta^{ij}$

$$\underset{4\times 4}{g} \cdot \underset{4\times 4}{g^{-1}} = \underset{4\times 4}{Id} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

<u>Homework</u>: Show that if g_{ij} transforms by $\bar{g}_{\alpha\beta} = \frac{\partial x^i}{\partial y^\alpha} g_{ij} \frac{\partial x^j}{\partial y^\beta}$ then g_{ij} is symmetric & invertible iff $\bar{g}_{\alpha\beta}$ is

(Hint: translate into matrices)

Theorem: $\langle X_p, Y_p \rangle$ is independent of coordinates iff g_{ij} transforms at each p as

$$g_{\alpha\beta} = \frac{\partial x^i}{\partial y^\alpha} g_{ij} \frac{\partial x^j}{\partial y^\beta}$$

$$\frac{\partial y^\alpha}{\partial x^i} g_{\alpha\beta} \frac{\partial y^\beta}{\partial x^j} = g_{ij}$$

Proof: Assume $X_p = a^i \frac{\partial}{\partial x^i}\Big|_p = \bar{a}^\alpha \frac{\partial}{\partial y^\alpha}\Big|_p$

$$Y_p = b^i \frac{\partial}{\partial x^i}\Big|_p = \bar{b}^\alpha \frac{\partial}{\partial y^\alpha}\Big|_p$$

so $a^i = \frac{\partial x^i}{\partial y^\alpha} \bar{a}^\alpha \quad b^j = \frac{\partial x^i}{\partial y^\alpha} \bar{b}^\alpha$. Then

$$\langle X_p, Y_p \rangle_x = a^i g_{ij} b^j = \bar{a}^\alpha \bar{g}_{\alpha\beta} \bar{b}^\beta = \underbrace{a^i \frac{\partial y^\alpha}{\partial x^i}}_{X_p} \bar{g}_{\alpha\beta} \underbrace{\frac{\partial y^\beta}{\partial x^j} b^j}_{Y_p}$$

$$= a^i \underbrace{\frac{\partial y^\alpha}{\partial x^i} \bar{g}_{\alpha\beta} \frac{\partial y^\beta}{\partial x^j}}_{g_{ij}} b^j$$

$\therefore$ Holds for all pairs of vectors iff

$$\boxed{g_{ij} = \frac{\partial y^\alpha}{\partial x^i} \bar{g}_{\alpha\beta} \frac{\partial y^\beta}{\partial x^j}}$$

$$(*) \qquad \underset{4\times4}{g} = \underset{4\times4}{J^T} \; \underset{4\times4}{\bar{g}} \; \underset{4\times4}{J}$$

mult thru by J^{-1} J^{-T}

$$\Leftrightarrow \quad \underset{4\times4}{J^{-T}} \; \underset{4\times4}{g} \; \underset{4\times4}{J^{-1}} = \underset{4\times4}{\bar{g}}$$

$$\boxed{\frac{\partial x^i}{\partial y^\alpha} g_{ij} \frac{\partial x^j}{\partial y^\beta} = \bar{g}_{\alpha\beta}}$$

As claimed ⊠

<u>Note</u>: Multiply (*) thru inverses –

$$J^{-1} \bar{g}^{-1} J^{-T} = g^{-1} \qquad g^{-1} = g^{ij}$$

$$\frac{\partial x^i}{\partial y^\alpha} \bar{g}^{\alpha\beta} \frac{\partial x^j}{\partial y^\beta} = g^{ij}$$

• Conclude: For $\int_{\underset{\sim}{x}(A)}^{\underset{\sim}{x}(B)} \|\dot{\underset{\sim}{x}}(\xi)\| d\xi$ to be independent of coordinates, it suffices to take $g_{ij}(\underset{\sim}{x})$ to be a symmetric non-degenerate $\binom{0}{2}$ tensor, which defines a bilinear form

$$\langle X_p, Y_p \rangle = g_{ij} a^i b^j = \bar{g}_{\alpha\beta} \bar{a}^\alpha \bar{b}^\beta = \langle X_p, Y_p \rangle_y$$

If $g_{ij} = \delta_{ij} \Rightarrow \; = \underset{\sim}{a} \cdot \underset{\sim}{b}$ = x-coord dot product

So $\|X_p\| = \sqrt{a^i g_{ij} a^j}$.

If tangent vector $X_p = \dot{x}^i \frac{\partial}{\partial x^i}\Big|_{p(\xi)}$ $\quad = \sqrt{\dot{x}^i g_{ij} \dot{x}^j}$

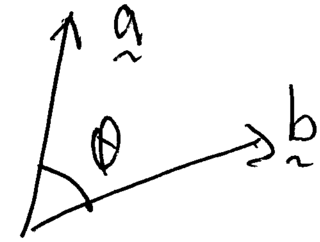
If $g_{ij} = \delta_{ij}$ $\quad = \sqrt{\sum_{i=0}^{3} (\dot{x}^i)^2}$ = x-coord Euclidean Length

- Just as dot product gives angles and lengths, so also does $\langle\ ,\ \rangle$

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = \|\underset{\sim}{a}\|\,\|\underset{\sim}{b}\|\cos\theta$$

↑ dot product on $\mathbb{R}^4$

$\mathbb{R}^4$



$$= a^i \delta_{ij} b^j$$

$$\langle X_p, Y_p\rangle = a^i g_{ij} b^j \ \text{“}=\text{”}\ \|\underset{\sim}{a}\|\,\|\underset{\sim}{b}\|\cos\theta$$

↗ "makes sense when translate into a basis that diagonalizes $g_{ij} \sim \delta_{ij}$"

- In GR we write:

$$\boxed{ds^2 = g_{ij}\, dx^i dx^j}$$

"arclength change for coord change dx^i along a curve"

Ie: $$L_{\underset{\sim}{x}(A)}^{\underset{\sim}{x}(B)} = \int_{\xi=a}^{\xi=b} ds = \int_a^b \sqrt{g_{ij}\, dx^i dx^j}$$

$dx^i = \dot{x}^i d\xi$ $$= \int_a^b \sqrt{g_{ij} \frac{dx^i}{d\xi} \frac{dx^j}{d\xi}}\, d\xi$$

$$= \int_a^b \| \dot{\underset{\sim}{x}}(\xi) \| \, d\xi \quad \checkmark$$

- Given $g_{ij}(\underset{\sim}{x})$ in a coord syst $\underset{\sim}{x}$

Q1: Are there coordinate systems in which General Relativity Reduces (locally) to special relativity?

Ans: Yes: $g_{ij}(\underset{\sim}{x}) = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} + \| \underset{\sim}{x} - \underset{\sim}{x}(P) \|^2$

"Locally Inertial Coordinates"

Q2: What measures the inability to remove the quadratic errors $|\underset{\sim}{x} - \underset{\sim}{x}(P)|^2$ from Special Relativity?

Ans: Riemann Curvature Tensor

$R^i_{\ jkl} \equiv \binom{1}{3}$ tensor.

Q3: What is the Einstein Curvature Tensor G in $G = \kappa T$

Ans: $$G_{ij} = R^{\sigma}_{i\sigma j} - \tfrac{1}{2} R\, g_{ij}$$

$$R^{\sigma}_{i\sigma j} = R^{0}_{i0j} + R^{1}_{i1j} + R^{2}_{i2j} + R^{3}_{i3j}$$

$\uparrow$

Sum repeated up-down indicer 0-3

Our Task: ~~Connect~~ (A) Disuss general $\binom{m}{n}$-tensors / How do they transform / how do you define them invariatly / Raising & Lowering of indices / contractions

(B) Discuss how you ~~get~~ recover Special Relativity in locally inertial frames