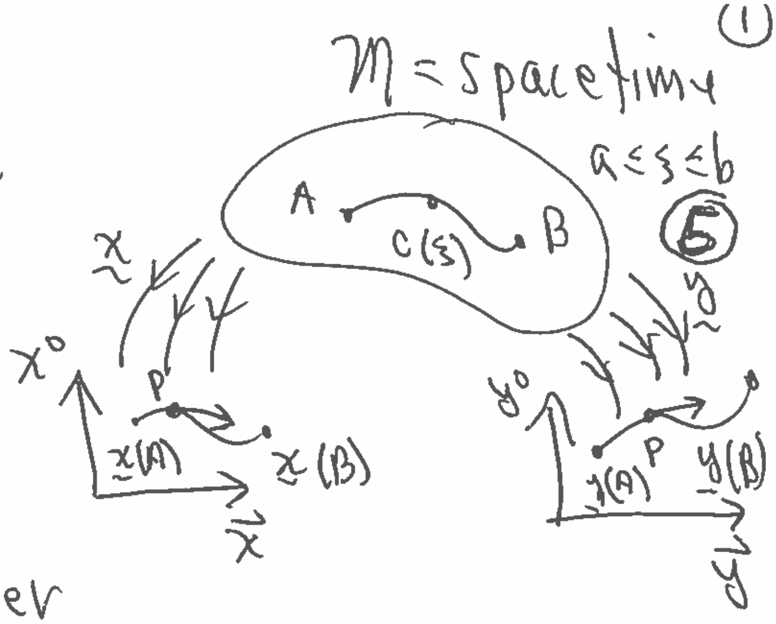
☒ Tensors: We have



– Summary –

- Spactime is the manifold of events $\mathcal{M}$

- Coordinate systems cover spacetime, and quantitative properties of spacetime are described in coord systems

- Curve $c(\xi)$ in spacetime can be represented in coordinate systems

$\underset{\sim}{x} \circ c(\xi) \equiv \underset{\sim}{x}(\xi)$ x-coord rep of c

$\underset{\sim}{y} \circ c(\xi) \equiv \underset{\sim}{y}(\xi)$ y-coord rep of c

Asking that tangent vectors to the same curve @ P, but represented in diff coord systems, be the same vector, tells us how to identify same vector in diff coords

- $T_pM \equiv$ tangent space of M @ p is the collection of all tangent vectors @ P, with representations of a vector $X_p \in T_pM$ in different coordinates viewed as the <u>same vector</u> given in terms of a different basis

$$\underbrace{a^i}_{\substack{x\text{-comps}\\ \text{of } X_p}} \underbrace{\frac{\partial}{\partial x^i}\bigg|_p}_{\substack{x\text{-coord}\\ \text{basis}}} = X_p = \underbrace{\bar{a}^\alpha}_{\substack{y\text{-comps}\\ \text{of } X_p}} \underbrace{\frac{\partial}{\partial y^\alpha}\bigg|_p}_{\substack{y\text{-coord}\\ \text{basis}}}$$

$$T_pM = \mathrm{Span}\left\{\frac{\partial}{\partial x^0}\bigg|_p, \dots, \frac{\partial}{\partial x^3}\bigg|_p\right\} = \left\{a^i \frac{\partial}{\partial x^i}\bigg|_p : a^i \in \mathbb{R}\right\}$$

$$= \mathrm{Span}\left\{\frac{\partial}{\partial y^0}\bigg|_p, \dots, \frac{\partial}{\partial y^3}\bigg|_p\right\} = \left\{\bar{a}^\alpha \frac{\partial}{\partial y^\alpha} : \bar{a}^\alpha \in \mathbb{R}\right\}$$

$$\boxed{a^i = \frac{\partial x^i}{\partial y^\alpha}\bar{a}^\alpha, \quad \frac{\partial}{\partial x^i}\bigg|_p = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha}\bigg|_p}$$

- $T_p^*M \equiv$ cotangent space of M @ p is the collection of all 1-forms @ p, each 1-form $\omega_p \in T_p^*M$ has a different representation wrt a different basis in each coordsystem.

$$\underbrace{a_i}_{\text{x-comp's of } \omega_p}\, \underbrace{dx^i\big|_p}_{\text{x-coord basis of 1-forms}} = \omega_p = \underbrace{\bar{a}_\alpha}_{\text{y-coord components of } \omega_p}\, \underbrace{dy^\alpha}_{\text{y-coord basis of 1-forms}}$$

1-forms are viewed as linear functionals on tangent vectors: $dx^i\big|_p : T_pM \longrightarrow \mathbb{R}$

$$dx^i(X_p) = a^i, \quad X_p = a^i \frac{\partial}{\partial x^i}$$

$$\omega_p(X_p) = a_i\, dx^i(X_p) = \underbrace{a_i b^i}_{X_p = b^i \frac{\partial}{\partial x^i}\big|_p} = \bar{a}_\alpha\, dy^\alpha(X_p) = \underbrace{\bar{a}_\alpha \bar{b}^\alpha}_{X_p = \bar{b}^\alpha \frac{\partial}{\partial y^\alpha}\big|}$$

$$\boxed{\begin{aligned} a_i &= \frac{\partial y^\alpha}{\partial x^i}\,\bar{a}_\alpha \\ dx^i\big|_p &= \frac{\partial x^i}{\partial y^\alpha}\, dy^\alpha\big|_p \end{aligned}}$$

- Conclude (Riesz Rep): $T_p^* M$ is the dual space of $T_p M$ (ie. the set of linear fn's on $T_p M$, "functional" means "maps to $\mathbb{R}$")
& $\dim T_p^* M = 4$.

$$T_p^* M = \mathrm{Span}\{dx^0|_p, \dots, dx^3|_p\} = \{a_i\, dx^i|_p \,|\, a^i \in \mathbb{R}\}$$

$$= \mathrm{Span}\{dy^0|_p, \dots, dy^3|_p\} = \{\bar{a}_\alpha\, dy^\alpha|_p, \, a^i \in \mathbb{R}\}$$

- General tensors like Riemann Curvature tensor $R^i{}_{jkl}$ have up & down indices; up transform like components of vectors a^i and down transform like component of covectors a_i. Natural generalization of a vector

<u>Key Point</u>: Transformation depends only on 1st derivative $J = \frac{\partial y^\alpha}{\partial x^i}$, <u>not</u> 2nd deriv's!

<u>Tensors</u>: A tensor is a multi-component matrix of up & down indices such that up indices transform contravariantly & down indices transform covariantly –

<u>Ex</u>: $R^{i}_{jkl} \equiv \binom{1}{3}$ - tensor $|_p$

$$R^{i}_{jkl} = \frac{\partial x^i}{\partial y^\alpha} R^{\alpha}_{\beta\gamma\delta} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} \qquad (*)$$

same tensor in y-coordinates

"You can think of tensors as multicomp matrices defined at a pt P such that they transform "linearly" in each T_pM by (*)

Operations on tensors —

- Raising & lowering by metric:

Ex: $R^{i}{}_{jkl}$ a $\binom{1}{3}$-tensor

$$R_{ijkl} = g_{im} R^{m}{}_{jkl} \text{ is a } \binom{0}{4}\text{-tensor}$$

$$R_{m}{}^{i}{}_{kl} = R^{i}{}_{jkl}$$

Thm: Lower & raising of indices is a tensor operation. $\boxed{\bar{g}^{\alpha\sigma} \bar{R}_{\sigma\beta\gamma\delta} = \frac{\partial y^\alpha}{\partial x^i} g^{ij} \frac{\partial y^\sigma}{\partial x^j} \frac{\partial x^k}{\partial y^\sigma} \underbrace{R_{k\beta\gamma\delta}}}$

$j = k$

- Contracting up-down indices

$$= \frac{\partial y^\alpha}{\partial x^i} \underbrace{g^{ik} R_{k\beta\gamma\delta}},$$

$$\frac{\partial y^\alpha}{\partial x^i} R^{i}{}_{\beta\gamma\delta},$$

Ex: $R^{m}{}_{iml} = R_{il}$

Thm: Contracting up down indices is a tensor operation

I.e. $\bar{R}^{\sigma}{}_{\alpha\sigma\beta} = \frac{\partial y^\sigma}{\partial x^i} R^{i}{}_{jkl} \frac{\partial x^j}{\partial y^\alpha} \frac{\partial x^k}{\partial y^\sigma} \frac{\partial x^l}{\partial y^\beta} = R^{k}{}_{jkl} \frac{\partial x^j}{\partial y^\alpha} \frac{\partial x^l}{\partial y^\beta}$ ✓

δ^{k}_{i} $k = i$

- Note: A metric gives meaning to the gradiant as a vector:

$$df = \frac{\partial f}{\partial x^i} dx^i \quad (\text{a 1-form} = \text{covector})$$

$$g^{ij} \frac{\partial f}{\partial x^i} = \partial^j f$$

Defn. We call $\partial^j f \frac{\partial}{\partial x^j}\Big|_p = \nabla f$ the gradiant of f at p

Note: if $g^{ij} = Id^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

then we can't tell differ Between ∇f & df !

- <u>Example</u>: Linear transformation:

Consider a matrix that defines a linear transformation of T_pM:

$$T : T_pM \longrightarrow T_pM$$

$$X_p \longmapsto Y_p$$

$$a^i \frac{\partial}{\partial x^i}\Big|_p \longrightarrow b^j \frac{\partial}{\partial x^j}\Big|_p$$

As matrix multiplication $T\underset{\sim}{a} = \underset{\sim}{b}$

$\swarrow$ row

As tensor: $T^{j}{}_{i}\, a^i = b^j$

$\nwarrow$ colm

In $\underset{\sim}{y}$-coords: $T^{j}{}_{i} \dfrac{\partial x^i}{\partial y^\alpha} \bar{a}^\alpha = \dfrac{\partial x^j}{\partial y^\beta} b^\beta$

$$\underbrace{\frac{\partial y^\beta}{\partial x^j} T^{j}{}_{i} \frac{\partial x^i}{\partial y^\alpha}}_{T^\beta{}_\alpha} \bar{a}^\alpha = b^\beta$$

$J^{-1} T_x J = T_y \iff T^\beta{}_\alpha \Rightarrow T$ is $\binom{1}{1}$-tensor

- Transformation matrices = $\binom{1}{1}$-tensors transform by similarity transformations
- Transformations have eigenvalues & eigenvectors defined indept of basis = coordinates:

Eg $\quad T^i_j a^j = \lambda a^i \qquad$ x-coords @ P

$\iff \quad T^\alpha_\beta \bar{a}^\beta = \lambda \bar{a}^\alpha \qquad$ y-coords @ P

$T : \underline{X}_p \longrightarrow \lambda \underline{X}_p \qquad$ indept of coords

(Homework)

• Q: T^i_j are the components of a $\binom{1}{1}$-tensor, so components wrt what basis? I.e.

$$X_p = \underbrace{a^i}_{\text{comps}} \underbrace{\frac{\partial}{\partial x^i}\Big|_p}_{\text{basis}}$$

$$T_p = \underbrace{T^i_j}_{\text{comps}} \cdot \underbrace{\frac{\partial}{\partial x^i} \otimes dx^j}_{\text{basis}}$$

$$\frac{\partial}{\partial x^i} \otimes dx^j \left[\omega_p, X_p\right] = a_i b^j$$

$$\omega_p = a_i dx^i \qquad X_p = b^j \frac{\partial}{\partial x^j}$$

"$\left\{\frac{\partial}{\partial x^i} \otimes dx^j\right\}$ is the x-coord basis for the multi-linear operators $L: (T^*_pM, T_pM) \to \mathbb{R}$"

This gives tensors a coord indept meaning.

- Thus we call $\binom{1}{1}_p M$ = set of all $\binom{1}{1}$-tensors at p.

$$\binom{1}{1}_p M = \text{Span}\left\{ \frac{\partial}{\partial x^i} \otimes dx^j \Big|_p \right\} = \left\{ A^i_j \frac{\partial}{\partial x^i} \otimes dx^j \Big|_p \right\}$$

$$= \text{Span}\left\{ \frac{\partial}{\partial y^\alpha} \otimes dy^\beta \Big|_p \right\} = \left\{ A^\alpha_\beta \frac{\partial}{\partial y^\alpha} \otimes dy^\beta \Big|_p \right\}$$

For example: $A \in \binom{1}{1}_p M \Rightarrow$

$$A = A^i_j \frac{\partial}{\partial x^i} \otimes dx^j \Big|_p : T^*_p M \times T_p M \longrightarrow \mathbb{R}$$

$$A[\omega_p, X_p] = A^i_j \frac{\partial}{\partial x^i} \otimes dx^j [\omega_p, X_p] = A^i_j a_i b^j$$

$\omega_p = a_i dx^i|_p$, $X_p = b^j \frac{\partial}{\partial x^j}\Big|_p$

Also:

$$A = A^\alpha_\beta \frac{\partial}{\partial y^\alpha} \otimes dy^\beta \Big|_p : T^*_p \times T_p M \to \mathbb{R}$$

(same operator — different basis)

$$A[\omega_p, X_p] = A^\alpha_\beta \bar{a}_\alpha \bar{b}^\beta = A^i_j a_i b_j$$ by tensor trans laws

- Conclude – Tensors are the natural linear objects that measure properties of spacetime at poin P –

Eg: Curvature / Velocity / Density / Energy Momentum vector of particle Energy-momentum density & their fluxes = stress energy tensor

Einstein – The equations that give the physical constraints on g_{ij} required for it to be a gravitational field should be expressed in terms of tensors –

Ans: $\boxed{G_{ij} = \kappa T_{ij}}$ ← An equation for g_{ij} given T_{ij}