☑ Special Relativity – Intro

- Gravitational metric tensor $g_{ij}(\underset{\sim}{x})$

A 4x4 symmetric, non-degenerate matrix given at every $\underset{\sim}{x}(p)$ of $\underset{\sim}{x}$-coords

$$\boxed{g_{\alpha\beta}(\underset{\sim}{y}) = \frac{\partial x^i}{\partial y^\alpha} g_{ij} \frac{\partial x^j}{\partial y^\beta}}$$

This determines a coordinate indept inner product:

$$\langle X_p, Y_p \rangle = \underbrace{g_{ij}\, a^i b^j}_{\text{compute in } x\text{-coords}} = \underbrace{\bar{g}_{\alpha\beta}\, \bar{a}^\alpha \bar{a}^\beta}_{\text{compute in } \underset{\sim}{y}\text{-coords}}$$

Thm: [Locally Inertial Frame Theorem]

- Given $g_{ij}(\underset{\sim}{x})$, $\underset{\sim}{x} = \underset{\sim}{x}(P)$, $P \in \mathcal{U}_x \subseteq \mathcal{M}$.
Then $\exists$ coordinate systems $\underset{\sim}{y}$ in a nbhd of P, such that, in $\underset{\sim}{y}$-coords

$$\left.\begin{aligned} & g_{\alpha\beta}(\underset{\sim}{y}) = \eta_{\alpha\beta} + \text{Error} \\ & |\text{Error}| \le C|\underset{\sim}{y} - \underset{\sim}{y}(P)|^2 \\ & \eta_{\alpha\beta} = \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \end{aligned}\right\} (*)_{ON}$$

- Moreover: The number of -1's & $+1$'s is the same at every point, so long as g_{ij} is symmetric & $\det g_{ij} \neq 0$.

In GR: We assume gravitational metrics are $(-1,1,1,1)$ metrics

$$\eta_{\alpha\beta} = \begin{bmatrix} -1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & +1 \end{bmatrix} \equiv \text{Minkowski Metric}$$

The coordinates (*) are called "locally inertial" or "locally Minkowski" and represent the closest you can get to Flat Spacetime = Minkowski Space

= Special Relativity

$$\left.\frac{\partial}{\partial y^0}\right|_p \equiv \text{timelike}$$

$$\left.\frac{\partial}{\partial y^i}\right|_p \equiv \text{spacelike}$$

- If we can make Error $\equiv 0$, $g \equiv \eta$, we say spacetime is <u>Flat</u>, or spacetime is <u>Minkowski</u> And then <u>Special Relativity</u> Holds.

- In general: we cannot make Error $= 0$ in any nbhd of P, in which case we say spacetime has non-zero Curvature at P.

- Riemann Curvature Tensor $- R^i_{jkl}$ measures the degree to which you cannot make Errors $= 0$.

<u>Thm.</u> If $R^i_{jkl} \equiv 0$, then $\exists\, \underset{\sim}{y}$ st $g_{\alpha\beta} \equiv \eta_{\alpha\beta}$
(Later)

- $(*)_{ON}$ is like saying that on a sphere you can always find coords where sphere looks like a plane to within 2nd order errors —

<u>Einstein</u>: Spacetime should be "locally Minkowski" in the sense that ∃ coordinate systems in which the laws of special relativity hold "to within 2nd order errors determined by the Curvature".

So this fits with his theory...

• <u>Conclude</u>: The assumption of general relativity, which follows <u>naturally</u> from the ~~assumption~~ observation that inertial mass = gravitational mass —

$$\underset{\text{inertial mass}}{\cancel{m}_p}\, \vec{a} = -G\, \underset{\text{gravitational mass}}{\cancel{m}_p}\, m_s \frac{1}{r^2} \frac{\vec{r}}{r}$$

$\rightsquigarrow$ gravity determines the <u>paths</u> not the <u>forces</u> ...

IS:

The gravitational field is a symmetric, non-degenerate metric g_{ij} ~~defined on~~ of signature $(-1,1,1,1)$ defined on spacetime

- All of the physical interpretations (like freefall paths, aging time, non-rotating frames in freefall, length/time etc) then follow by <u>simply</u> assuming special relativity holds in each locally inertial frame (to within 2nd order errors). I.e.,

freefall paths = "straight lines in locally inertial frame"
$\Rightarrow$ geodesics of g_{ij}

proper time = aging time = "aging time in locally inertial frames"
$\Rightarrow$ arclength of g_{ij}

non-rotating frames = "non-rotating relative to locally inertial frame" $\Rightarrow$ <u>Connection</u>

- Only one missing element —
  What equation must g_{ij} satisfy
  in order to be a gravitational field?

Ans: Einstein Equation: $\boxed{G[g] = \frac{8\pi G}{c^4} T}$

equation for g
involving 2nd
derivatives