Special Relativity: (Details) (8)

- Assumption of GR: Gravitational Field is described by a symmetric non-degenerate $\binom{0}{2}$-tensor $g_{ij}\, dx^i \otimes dx^j$ of signature $(-1,1,1,1) \Rightarrow$ by locally inertial frames

Thm about each point P & coord system $\underset{\sim}{x}$ defined in a nbhd of P, $\exists$ coord system $\underset{\sim}{y}$ in which ($\underset{\sim}{y} \equiv$ orthonormal @ P)

$$g_{\alpha\beta}(\underset{\sim}{y}) = \eta_{\alpha\beta} + \underbrace{O(|\underset{\sim}{y} - \underset{\sim}{y}(P)|^2)}$$

$$\eta_{\alpha\beta} = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}.$$

Contains info about curvature of spacetime.

Special Relativity is case when $\exists$ coord system in which $g_{\alpha\beta} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ everywhere

- Gravitational metric:

$$ds^2 = g_{ij}\, dx^i dx^j \equiv g_{ij}\, dx^i \otimes dx^j$$

Length of Curve: $L_A^B = \int_{\xi=a}^{\xi=b} ds = \int_{\xi=a}^{\xi=b} \sqrt{g_{ij}\,\dot{x}^i \dot{x}^j}\, d\xi$

$$= \int_{\xi=a}^{\xi=b} \| X(\xi) \|\, d\xi$$

indep of coords

$$X(\xi) = \sqrt{|g_{ij}\,\dot{x}^i \dot{x}^j|} = \sqrt{|g_{\alpha\beta}\,\dot{y}^\alpha \dot{y}^\beta|} = \sqrt{|\langle X, X \rangle|}$$

Inner product:

$$\langle X_p, Y_p \rangle = g_{ij}\, a^i b^j$$

$$X_p = a^i \frac{\partial}{\partial x^i}\Big|_p \quad Y_p = b^\alpha \frac{\partial}{\partial y^\alpha}\Big|_p$$

Since

- Since $\operatorname{sng}(g) = (-1, 1, 1, 1)$, inner product can be neg or positive —

$\langle X_p, X_p \rangle < 0 \iff X_p$ timelike

$\langle X_p, X_p \rangle = 0 \iff X_p$ lightlike

$\langle X_p, X_p \rangle > 0 \iff X_p$ spacelike

Note that whether X_p time/light/spacelike can be determine in the ON frames

I.e. $X_p = a^i \frac{\partial}{\partial x^i}\Big|_p$, $g_{\alpha\beta} \equiv \begin{bmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$

$$\langle X_p, X_p \rangle = a^i g_{ij} a^j = a^i \frac{\partial y^\alpha}{\partial x^i} g_{\alpha\beta} \frac{\partial y^\beta}{\partial x^j} b^j$$

$$= \underbrace{\left(a^i \frac{\partial y^\alpha}{\partial x^i}\right)} \begin{bmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}_{\alpha\beta} \underbrace{\left(\frac{\partial y^\beta}{\partial x^j} b^j\right)}$$

y-comps of X_p

(4a)

$$\frac{\partial}{\partial y_0} + \sum_{k=1}^{3} a^k \frac{\partial}{\partial x^k} = 0 \quad \text{iff} \quad a^k \cdot a^k = 1 = \vec{a} \cdot \vec{a}$$

$$(1, a^1, a^2, a^3) \begin{bmatrix} -1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \begin{pmatrix} 1 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = -1 + \vec{a} \cdot \vec{a} = 0$$

$$\text{iff} \quad \vec{a} \cdot \vec{a} = 1$$

Lightlike vectors lie on cone

$$-(a^0)^2 + \sum_{k=1}^{3} (a^k)^2 = 0$$

~~Conclude: The on frames determine whether $\|X_p\|$~~

<u>Conclude</u>: The ON. frames $\underset{\sim}{y}$ determine whether $\underline{X}_p$ timelike / spacelike / timelike

$\frac{\partial}{\partial y_0}$ timelike $\quad \left\langle \frac{\partial}{\partial y_0}, \frac{\partial}{\partial y_0} \right\rangle = (1,0,0,0)\begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$= -1$$

$\frac{\partial}{\partial y_i}$ spacelike eg. $\left\langle \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_2} \right\rangle = (0,0,1,0)\begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$= +1$$

Principle of GR: GR reduces to special relativity in ON frames (to within errors $\sim O(|\underset{\sim}{y} - \underset{\sim}{y}(p)|^2)$)

Special Relativity: There exists a global coord system (call it $\underset{\sim}{x}$) in which

$$g_{ij}(\underset{\sim}{x}) = \eta_{ij} = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}.$$

- We say $x^0 = ct$, c = speed of light, t in seconds, x^i in meters $i = 1, 2, 3$

so $[x^i]$ = meters $i = 0, 1, 2, 3$

- The <u>world line</u> of a particle is its trajectory $\underset{\sim}{x}(s)$ in $\underset{\sim}{x}$-coordinates

- Physical Assumptions of Special Relativity

(A1) The world lines of particles always have <u>timelike</u> tangent vectors ≡ "physical object move at < speed of light"

(A2) The aging time = proper time change for observers traversing a world line is the |length of curve| (same in GR as Spec. Rel)

$$ds^2 = g_{ij}\, dx^i dx^j \qquad \text{(length)}$$

$$\boxed{\xi = c\tau} \qquad c^2 d\tau^2 = g_{ij}\, dx^i dx^j$$

$$\Delta\tau = \frac{1}{c^2} \int_{\xi=a}^{\xi=b} \sqrt{\underbrace{g_{ij}\dot{x}^i \dot{x}^j}_{\substack{\text{neg for} \\ \text{timelike vectors} \\ \dot{x}^i \frac{\partial}{\partial x^i}}}}\, d\xi = \frac{1}{c^2} \int_{\xi=a}^{\xi=b} \left\| - \frac{dx}{d\xi} \right\| d\xi$$

proper time in seconds

<u>Note</u>: $g_{ij} = \eta_{ij} = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$

Observor fixed in $\underset{\sim}{x}$-coords moves along curve $x^0 = \xi$, $x^i = 0 \Rightarrow$ tangent vector $\frac{\partial}{\partial x^0} \Rightarrow$

$$\Delta S = \int_0^{\xi} \sqrt{-\eta_{ij}\dot{x}^i\dot{x}^j}\, d\xi = \int_0^{\xi} \sqrt{-\eta_{00}}\, d\xi = \xi$$

$$= x^0$$

<u>Conclude</u>: Observors fixed wrt (x^1, x^2, x^3) in an ON frame $\underset{\sim}{x}$ age according to their time coordinate x^0.

"proper time = coord time in ON frame fixed with observor"

(Coord time $\neq$ proper time in moving frame)

(A3) For spacelike curves at $x^0 = 0$, The spatial length of curve is given by metric:

$$x^0 = 0 \,, \quad x^i = x^i(\xi) \quad i = 1,2,3$$

$$\Delta s = \int_{\xi_1}^{\xi_2} \sqrt{-\eta_{ij}\,\dot{x}^i \dot{x}^j}\, d\xi = \int_{\xi_1}^{\xi_2} \sqrt{(\dot{x}^1(\xi))^2 + (\dot{x}^2(\xi))^2 + (\dot{x}^3(\xi))^2}\, d\xi$$

$$= \int_{\xi_1}^{\xi_2} \sqrt{\dot{\vec{x}}(\xi) \cdot \dot{\vec{x}}(\xi)}\, d\xi = \text{Euclidean length of curve}$$

(A4) Free fall paths are straight lines in Minkowski space $g_{ij} = \eta_{ij}$, and non-rotating vectors along free fall paths are "parallel transported by the coordinate"

Ie $\quad a^i(\xi) \frac{\partial}{\partial x^i}\Big|_{c(\xi)} \parallel$ along $c(\xi)$

iff $a^i(\xi) = \text{const}$.

X spacelike: $ds(X)$ is the length in meters of a rod as measured by observer moving in a frame in which P_0 and P_1 occur at same time. E.g., in $\underset{\sim}{x}$-frame, $X^0 = 0 \Rightarrow ds(X) = \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2}$ is pos def metric giving Euclidean lengths.

- Property of Flat Space: ① You can identify vectors in T_pM with T_qM by:

$$g_{ij} \equiv \eta_{ij} \text{ in coords } \underset{\sim}{x}^i$$

$$\Leftrightarrow \quad X_p = a^i \frac{\partial}{\partial x^i}\Big|_p \longleftrightarrow X_q = a^i \frac{\partial}{\partial x^i}\Big|_q$$

"vectors with same components at different pts are said to be ∥-translations"

② You can identify T_0M with M by:

$$X = a^i \frac{\partial}{\partial x^i}\Big|_{\underset{\sim}{x}=0} \longleftrightarrow x^i = a^i \equiv \underset{\sim}{x} \text{ coords of a pt in } M$$

Define: $X \in T_pM$, $\Phi(X) = q \in M : \underset{\sim}{x}^i(q) - \underset{\sim}{x}^i(p) = X^i$

Example : 1-d $\qquad g_{ij} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

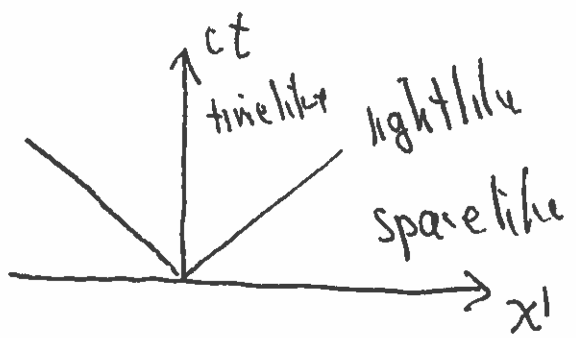
Check: $\left\{ \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1} \right\}$ form an O-N basis

$$\left\langle \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0} \right\rangle = g_{ij}\, e_0^i\, e_0^j = -1$$

$$\left\langle \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1} \right\rangle = +1$$

$$\left\langle \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1} \right\rangle = 0$$

Check: $\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}$ lightlike $\quad [1, \pm 1] \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = 0$



Q: What are the other O-N frames, and how are they related to $\underline{x}$-coordinates?

Ans: Let $X = x^0 \frac{\partial}{\partial x^0} + x^1 \frac{\partial}{\partial x^1}$

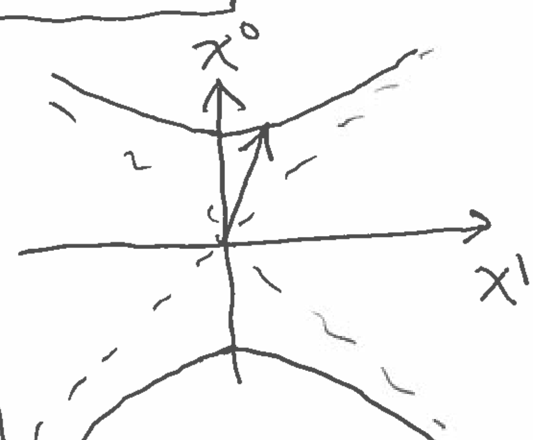
identify comp's of vectors with points in the coord system

$$\langle X, X \rangle = -1 \iff (x^0, x^1) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

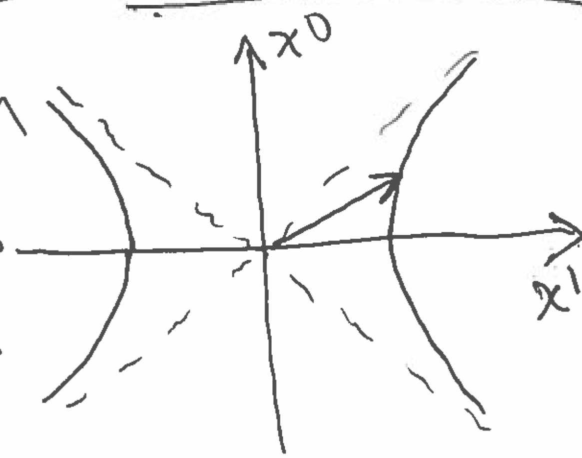
$$\iff -(x^0)^2 + (x^1)^2 = -1$$

$$\iff \boxed{(x^0)^2 - (x^1)^2 = 1}$$

Points toward pos time $\Rightarrow$ on the <u>upper</u> hyperbola



$$\langle X, X \rangle = 1 \iff \boxed{(x^1)^2 - (x^0)^2 = 1}$$



Only unit spacelike vectors on Right hand hyperbola are pos oriented wrt pos timelike vectors

$\{\underline{X}_0, \underline{X}_1\}$ form an ON basis iff

$\langle \underline{X}_0, \underline{X}_0 \rangle = -1$, $\langle \underline{X}_1, \underline{X}_1 \rangle = 1$, $\langle \underline{X}_0, \underline{X}_1 \rangle = 0$

Timelike unit   spacelike unit

$$\underline{X}_0 = a^0 \frac{\partial}{\partial x^0} + a^1 \frac{\partial}{\partial x^1} \qquad \underline{X}_1 = b^0 \frac{\partial}{\partial x^0} + b^1 \frac{\partial}{\partial x^1}$$

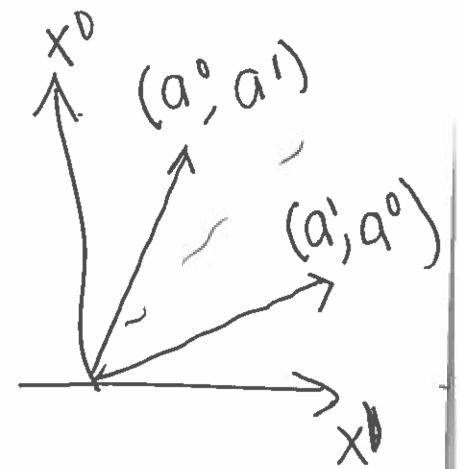
$$\langle \underline{X}_0, \underline{X}_1 \rangle = 0 \iff -a^0 b^0 + a^1 b^1 = 0$$

$$(+a^0, a^1) \cdot (-b^0, b^1) = 0$$

dot prod of $\mathbb{R}^2$

$$\iff (-b^0, b^1) \parallel (-a^1, a^0)$$

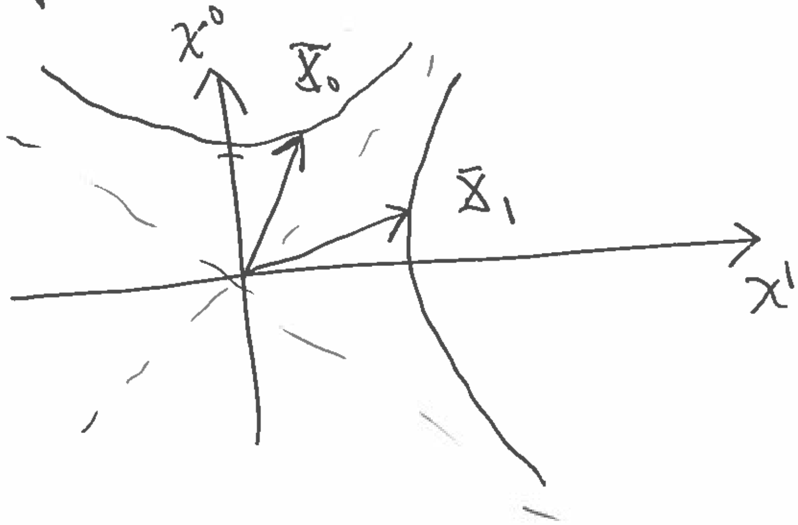
$$\iff (b^0, b^1) \parallel (a^1, a^0)$$



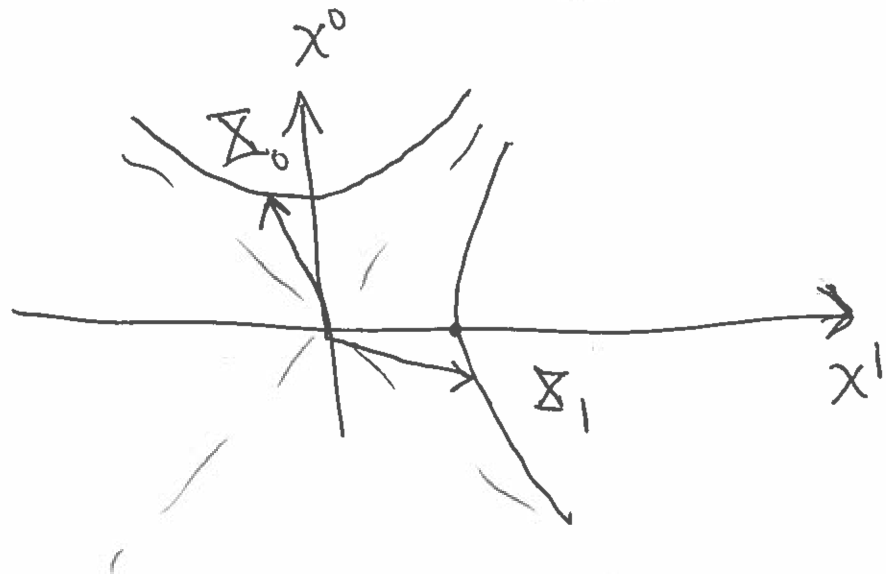
<u>Conclude</u>: $\langle \underline{X}_0, \underline{X}_1 \rangle = 0$ iff

$\underline{X}_1$ is on the reflection of $\underline{X}_0$ in $45°$ line

<u>Conclude</u>: $\langle \bar{X}_0, \bar{X}_1 \rangle$ is pos oriented ON frame with $\bar{X}_0$ pointing toward pos time iff



or



Note: You cannot get from pos time pos oriented ON frame by a ~~smooth~~ continuous change in ON. frame $\{\bar{X}_0, \bar{X}_1\}$.

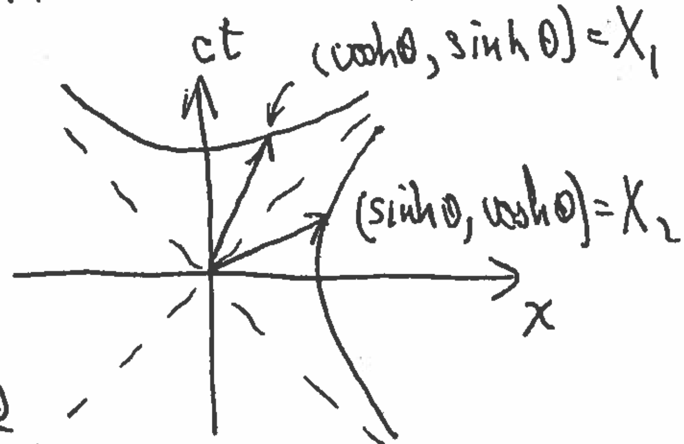
Let $x^1 = x$, $x^0 = ct$

$\cosh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$

$\sinh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$

Note: $\cosh^2\theta - \sinh^2\theta = 1$

$\Rightarrow ct = \cosh\theta$, $x = \sinh\theta$ lies on unit hyperbola. Our notation (a, b) $a = x^0$-coord, $b = x^1$-coord



$$\Rightarrow X_0 = \cosh\theta \frac{\partial}{\partial x^0} + \sinh\theta \frac{\partial}{\partial x^1}$$

$$X_1 = \sinh\theta \frac{\partial}{\partial x^0} + \cosh\theta \frac{\partial}{\partial x^1}$$

gives all pos oriented, time oriented, o.n. frames, $-\infty < \theta < \infty$.

Note: ① $\forall$ X_0 with $|X_0| = 1$, you can complete it to an O.N. frame. If further X_0 is pos-time directed, then you can complete it uniquely to frame (X_0, X_1) with X_1 pos. space directed.

② All vectors can be completed to o.n frame except lightlike vectors, which are $\perp$ to themselves.

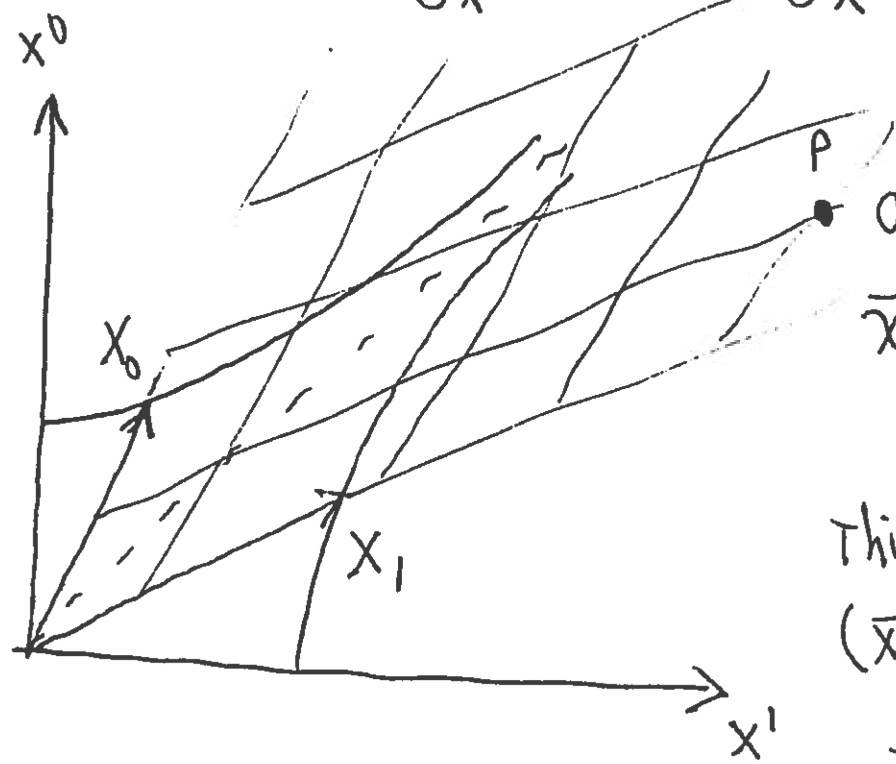
- <u>Lorentz Transformations</u>: Given $g_{ij} = \eta_{ij}$ in $\underset{\sim}{x} \equiv (x^0, x^1)$ coordinates. Construct another O-N frame for each

$$X_0 = \cosh\theta \frac{\partial}{\partial x^0} + \sinh\theta \frac{\partial}{\partial x^1}$$

$$X_1 = \sinh\theta \frac{\partial}{\partial x^0} + \cosh\theta \frac{\partial}{\partial x^1}$$

I.e., translate these vectors to each point of spacetime ("‖-translation in a flat spacetime") and choose $\{X_0, X_1\}$ to be the coord. basis vectors for a new coord system $(\bar{x}^0, \bar{x}^1)$ on spacetime as follows: We need

$$X_0 = \frac{\partial}{\partial \bar{x}^0}, \quad X_1 = \frac{\partial}{\partial \bar{x}^1}$$



$aX_0 + bX_1$

$\bar{x}^0(P) = a, \ \bar{x}^1(P) = b$

↑

This defines the $(\bar{x}^0, \bar{x}^1)$ coord system.

Clearly, $\frac{\partial}{\partial \bar{x}^0} = X_0$, $\frac{\partial}{\partial \bar{x}^1} = X_1$ FIP, and

thus $\bar{g}_{ij} = \eta_{ij}$ because $\{X_0, X_1\}$ is an O-N basis at each point of spacetime.

Q: How is the (x^0, x^1) coord. system related to the $(\bar{x}^0, \bar{x}^1)$ coord system?

Ans: $x^0 \frac{\partial}{\partial x^0} + x^1 \frac{\partial}{\partial x^1} = \bar{x}^0 \frac{\partial}{\partial \bar{x}^0} + \bar{x}^1 \frac{\partial}{\partial \bar{x}^1}$

$\Leftrightarrow$ "coord's name same point p"

But: $$\frac{\partial}{\partial \bar{x}^0} = \cosh\theta \frac{\partial}{\partial x^0} + \sinh\theta \frac{\partial}{\partial x^1}$$

$$\frac{\partial}{\partial \bar{x}^1} = \sinh\theta \frac{\partial}{\partial x^0} + \cosh\theta \frac{\partial}{\partial x^1}$$

$$\Rightarrow x^0 \frac{\partial}{\partial x^0} + x^1 \frac{\partial}{\partial x^1} = (\bar{x}^0 \cosh\theta + \bar{x}^1 \sinh\theta)\frac{\partial}{\partial x^0} + (\bar{x}^0 \sinh\theta + \bar{x}^1 \cosh\theta)\frac{\partial}{\partial x^1}$$

$$\Leftrightarrow \begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} \bar{x}^0 \\ \bar{x}^1 \end{bmatrix}$$

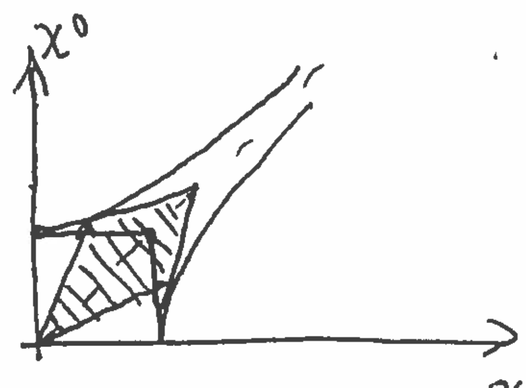
Theorem: The positively oriented, time oriented, homogeneous Lorentz transformations are given by $\underset{\sim}{x} = L(\theta)\underset{\sim}{\bar{x}}$, where $L(\theta)$ is given by

$$\begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = L(\theta)\begin{bmatrix} \bar{x}^0 \\ \bar{x}^1 \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}\begin{bmatrix} \bar{x}^0 \\ \bar{x}^1 \end{bmatrix}$$

for $-\infty < \theta < \infty$.

Note ①: $\det L(\theta) = \cosh^2\theta - \sinh^2\theta = 1 \Rightarrow$ Lorentz transformations preserve the coordinate volume:

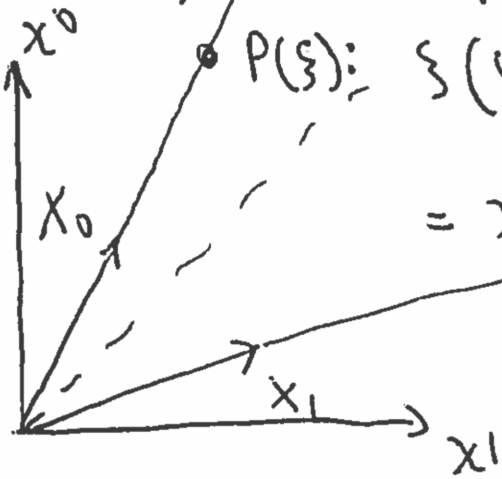
I.e., Vol of  $= 1$.



Note ②: $L(\theta)^{-1} = L(-\theta) = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix}$

Note ③: Lightlike vectors move with speed $\frac{dx}{dt} = c$ in every ON frame ⇒ Speed of light constant in every frame

Note ④ we can rewrite in terms of the velocity of the $\bar{x}$-frame as observed in the x-frame by writing $\begin{Bmatrix}\cosh\theta \\ \sinh\theta\end{Bmatrix}$ as a fn of v:



$$P(\xi):\ \xi\left(\cosh\theta \frac{\partial}{\partial x^0} + \sinh\theta \frac{\partial}{\partial x^1}\right)$$
$$= x^0(\xi)\frac{\partial}{\partial x^0} + x^1(\xi)\frac{\partial}{\partial x^1}$$

Conclude: $P(\xi)$ parameterizes the $\bar{x}^0$-axis

$$\Leftrightarrow \quad \frac{dx^1}{dx^0} = \frac{dx^1/d\xi}{dx^0/d\xi} = \frac{\sinh\theta}{\cosh\theta} \qquad \begin{cases} x^0(\xi) = \xi\cosh\theta \\ x^1(\xi) = \xi\sinh\theta \end{cases}$$

$$\therefore \quad \frac{dx^1}{dx^0} = \frac{1}{c}\frac{dx^1}{dt} = \frac{1}{c}v = \frac{\sinh\theta}{\cosh\theta} = \tanh\theta$$

$$1 - \tanh^2\theta = \operatorname{sech}^2\theta = \frac{1}{\cosh^2\theta} \Rightarrow \cosh^2\theta = \frac{1}{1-\left(\frac{v}{c}\right)^2}$$

$$-1 + \tanh^{-2}\theta = \operatorname{csch}^2\theta = \frac{1}{\sinh^2\theta} \Rightarrow \sinh^2\theta = \frac{\left(\frac{v}{c}\right)^2}{1-\left(\frac{v}{c}\right)^2}$$

$\Rightarrow$ Lorentz Transformation

$$\begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1-(v/c)^2}} & \frac{v/c}{\sqrt{1-(v/c)^2}} \\ \frac{v/c}{\sqrt{1-(v/c)^2}} & \frac{1}{\sqrt{1-(v/c)^2}} \end{bmatrix} \begin{bmatrix} \bar{x}^0 \\ \bar{x}^1 \end{bmatrix}$$

gives L. trans, where the bar frame moves with vel. v rel to unbarred frame.

Here, $\sqrt{1-\left(\frac{v}{c}\right)^2} = 1 - \frac{1}{2}\left(\frac{v}{c}\right)^2 + O\left(\frac{v}{c}\right)^4$

$$\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1}{1-\frac{1}{2}\{\ \}} = 1 + \frac{1}{2}\{\ \} + O(\{\ \})^2$$
$$= 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + O\left(\frac{v}{c}\right)^4$$

$$\Rightarrow \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + O\left(\frac{v}{c}\right)^4$$

$\Rightarrow$ $L(\theta) =$

$\Rightarrow$ neglecting $O\left(\frac{v}{c}\right)^2$

$$x^0 = \bar{x}^0 + \frac{v}{c}\bar{x}^1 \iff ct = c\bar{t} + \frac{v}{c}\bar{x}^1 \iff t = \bar{t} + O\left(\frac{1}{c^2}\right) \quad v = O(1),\ \bar{x}^1 = O(1)$$

$$x^1 = \frac{v}{c}\bar{x}^0 + \bar{x}^1 \iff x^1 = v\bar{t} + \bar{x}^1 \iff x^1 = v\bar{t} + \bar{x}^1$$

$\approx$ Galilean Trans

- <u>Time</u> <u>Dialation</u>: An observor fixed in the unbarred frame (say at origin) moves along a curve $x^1 = 0$, $x^0 = \xi$. His "aging time" betw t_1 and t_2 is given by

$$c\Delta\tau = \int_{ct_1}^{ct_2} \sqrt{|-\eta_{ij}\dot{x}^i \dot{x}^j|}\, d\xi = \int_{ct_1}^{ct_2} d\xi = c\Delta t = \Delta x^0$$

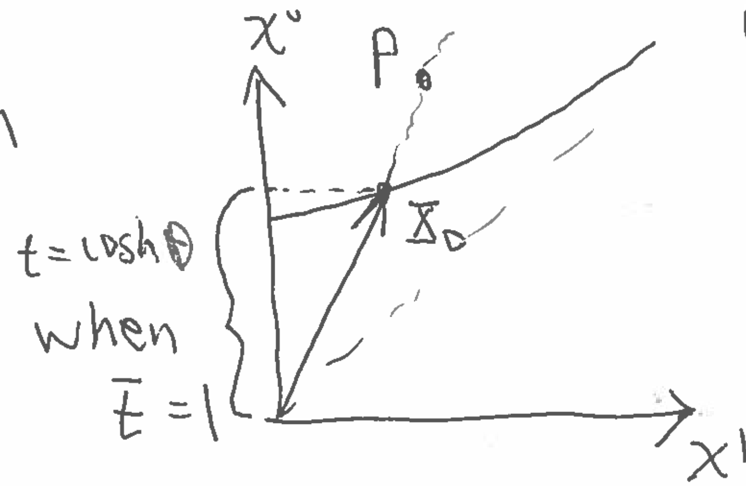
<u>Conclude</u>: Proper time & coordinate time agree for an observer fixed in L-frame.

<u>By symmetry</u>, an observer fixed on $\bar{x}^0$-coord axis ages according to the change in his $\bar{t}$-coordinate.

Time Dilation:

But: $\bar{t} = 1$ when

$t = \cosh\theta$



$$\frac{t(P)}{\bar{t}(P)} = \cosh\theta = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} > 1$$

"Moving clocks appear to run slower"

HW Twin Paradox: Any observer that moves and returns will age less than a fixed observer. Explain lack of symmetry.

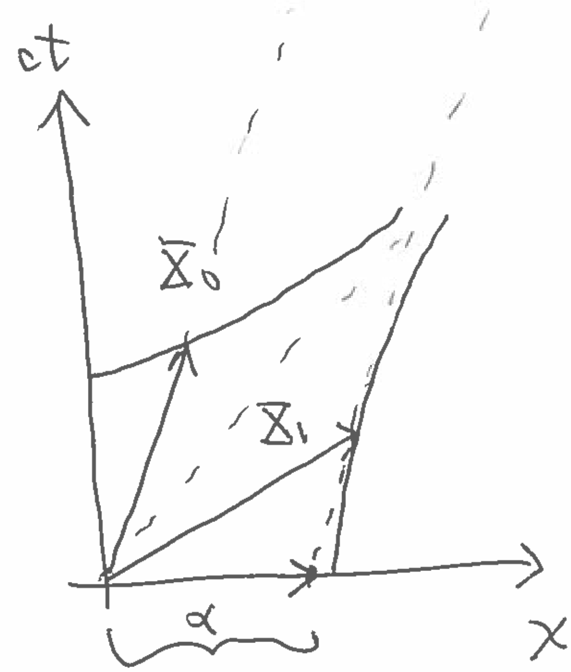
HW Show $L(\theta)L(\bar{\theta}) = L(\theta+\bar{\theta})$ and use it to derive the relativistic addition of velocity formula $\bar{\bar{v}} = \dfrac{v+\bar{v}}{1+\frac{v\bar{v}}{c^2}}$

where v is vel of $\bar{x}$ wrt x, and $\bar{v}$ vel of $\bar{\bar{x}}$ wrt $\bar{x}$.

Lorentz Contraction:

Consider a moving rod fixed in $\bar{x}$-coords:

α = length of unit rod as measured in x-coords



"Length of rod" in x-coords when $\bar{x} = 1$ in $\bar{x}$-coordinates

$$\alpha \frac{\partial}{\partial x} + \beta \bar{X}_0 = \bar{X}_1$$

Take inner product of both sides wrt $\bar{X}_1$ & use

$$\langle \bar{X}_0, \bar{X}_1 \rangle = 0, \quad \langle \bar{X}_1, \bar{X}_1 \rangle = 1$$

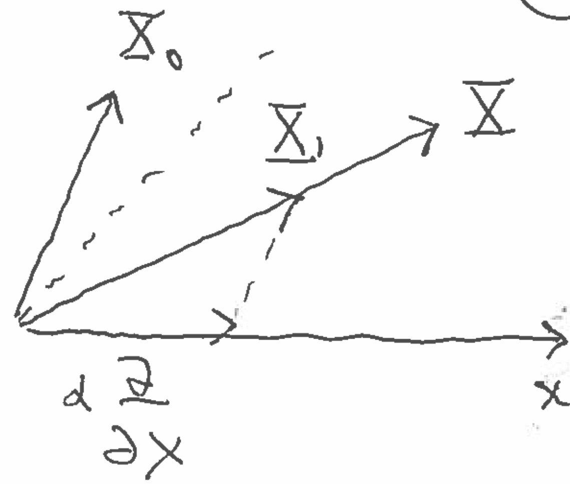
$$\alpha \underbrace{\left\langle \frac{\partial}{\partial x}, \bar{X}_1 \right\rangle}_{x\text{-comp of } \bar{X}_1 \,=\, \cosh\theta} = 1$$

$$\alpha = \frac{1}{\cosh\theta}$$

$$\therefore \quad \frac{L_x}{L_{\bar{x}}} = \alpha = \cosh^{-1}\theta \qquad L_x = \cosh^{-1}\theta \, L_{\bar{x}}$$

$$L_x = \sqrt{1 - \left(\frac{v}{c}\right)^2} \, L_{\bar{x}} \iff$$ "moving rods appear contracted"

Looking at this more carefully, we can say that the orthogonal projection of $\alpha \frac{\partial}{\partial x}$ onto $\mathbf{X}$ is $\mathbf{X}_1$.



$$\operatorname{Proj}_{\mathbf{X}} \alpha \frac{\partial}{\partial x} = \mathbf{X}_1$$

More generally, let $\mathbf{X}$ be any non-light like vector $\langle \mathbf{X}, \mathbf{X} \rangle \neq 0$, and let $\hat{\mathbf{X}}$ be orthogonal to $\mathbf{X}$, so $\langle \hat{\mathbf{X}}, \mathbf{X} \rangle = 0$. ($\hat{\mathbf{X}} \neq c\mathbf{X}$ exists iff $\langle \mathbf{X}, \mathbf{X} \rangle \neq 0 \Rightarrow \{\hat{\mathbf{X}}, \mathbf{X}\}$ is a basis)

Given any othe vector $\mathbf{Y}$, we know $\exists \alpha, \beta$ st

$$\mathbf{Y} = \alpha \mathbf{X} + \beta \hat{\mathbf{X}}$$

Take inner product with $\mathbf{X}$ $\Rightarrow$

$$\langle \mathbf{Y}, \mathbf{X} \rangle = \alpha \langle \mathbf{X}, \mathbf{X} \rangle + \beta \cancelto{0}{\langle \hat{\mathbf{X}}, \mathbf{X} \rangle}$$

$$\therefore \alpha = \frac{\langle \bar{Y}, \bar{X} \rangle}{\langle \bar{X}, \bar{X} \rangle}$$

$$\Rightarrow \boxed{\text{Proj}_{\bar{X}} \bar{Y} = \frac{\langle \bar{Y}, \bar{X} \rangle}{\langle \bar{X}, \bar{X} \rangle} \bar{X}}$$

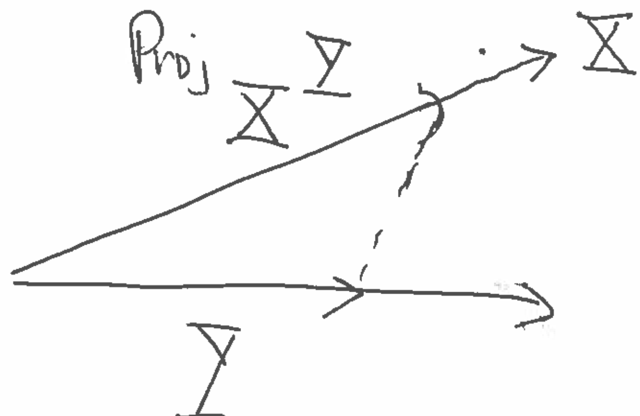
± sign keep track of relativity

$$\boxed{\text{Proj}_{\bar{X}} \bar{Y} = \frac{\langle \bar{Y}, \bar{X} \rangle}{\langle \bar{X}, \bar{X} \rangle} \bar{X}}$$

• We can use the projection to construct an ON basis out of any 4 non-lightlike vectors —

i.e. given $\{\underline{X}, \underline{Y}\}$,

$\{(\underline{Y} - \text{Proj}_{\underline{X}}\, \underline{Y}), \underline{X}\}$ are



orthogonal, so scale to unit length to get ON basis. Extend to 4-D as in Euclidean Case. FIP)

Special Relativity in (3+1): Assume $g_{ij} = \eta_{ij} = \mathrm{diag}(-1,1,1,1)$ in $\underset{\sim}{x}$-coords $\Leftrightarrow$ $\underset{\sim}{x}$ a Lorentz frame. (1)

Defn: A Lorentz transformation is a map between two coord. systems in which $g = \eta$.

Condition: $$\eta_{\alpha\beta} = \eta_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\alpha}$$

$\Leftrightarrow$ Matrix notation: $A = \dfrac{\partial x^i}{\partial y^\alpha}$ $\begin{array}{l} \leftarrow \text{row} \\ \leftarrow \text{colum} \end{array}$

$$\eta = A^t \eta A \qquad \text{(L)}$$

Notation: $x \equiv (x^0, \dots, x^3)$ $\underset{\sim}{x} \equiv (x^1, x^2, x^3) \in \mathbb{R}^3$

☒ Main Steps —

- Thm ①: If (L) holds at every point, then $x \circ y^{-1}$ is a linear transformation

$$(x \circ y^{-1})(y) = \underset{4\times 4}{A}\, y + \underset{4\times 1}{a} \qquad (*)$$

where $\det A = 1$, A a constant matrix

- ~~Thm~~ ② All transformations $(*)$ forms the inhomogeneous Lorentz group or Poincare group of transformations of spacetime. If $a=0$, its the homogeneous Lorentz group. If $\det A > 0$ & pos time preserving, then the proper (homogeneous) Lorentz group

- **Defn**: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{bmatrix}$ where

R is a 3×3 rotation, then $x \circ y^{-1}$ is called a rotation

- ~~Thm: If~~ If A changes velocity by $\vec{v}$, then A is called a boost

- **Thm**: If $A, \bar{A}$ have same velocity then $A = \bar{A} R$ $\boxed{v^i = c \dfrac{A^i_0}{A^0_0}}$

- **Thm**: if you know one boost $A_{\vec{v}}$, then all A's are given by

$$A = A_{\vec{v}} R \qquad (\vec{v} \equiv \underset{\sim}{v})$$

(Characterizes all the A's)

Thm. A canonical boost —

$$A = \begin{bmatrix} \gamma & -\gamma \left(\frac{\underset{\sim}{v}}{c} \right)^T \\ -\gamma \frac{\underset{\sim}{v}^T}{c} & I + (\gamma - 1) \frac{\underset{\sim}{v}^T \underset{\sim}{v}}{|\underset{\sim}{v}|^2} \end{bmatrix}$$

$$\boxed{V^i = c \, \frac{A^i_0}{A^0_0}} \qquad \gamma = \frac{1}{\sqrt{1 - \frac{|\underset{\sim}{v}|^2}{c^2}}}$$