Connection – Covariant Derivative

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- Einstein's Assumption – Spacetime is "Locally Lorentzian" so special relativity holds in locally inertial coordinates (to within 2nd order errors)

I.e., the assumption of GR is that gravity is described by a metric tensor $g_{ij}dx^i \otimes dx^j$ of signature $(-1,1,1,1)$ so —

<u>Thm</u>: $\forall$ $p \in M$ $\exists$ coord system $\underset{\sim}{x}$ in which $\underset{\sim}{x}(p)=0$ and

$$g_{ij}(\underset{\sim}{x}) = \eta_{ij} + O(|\underset{\sim}{x}|^2)$$

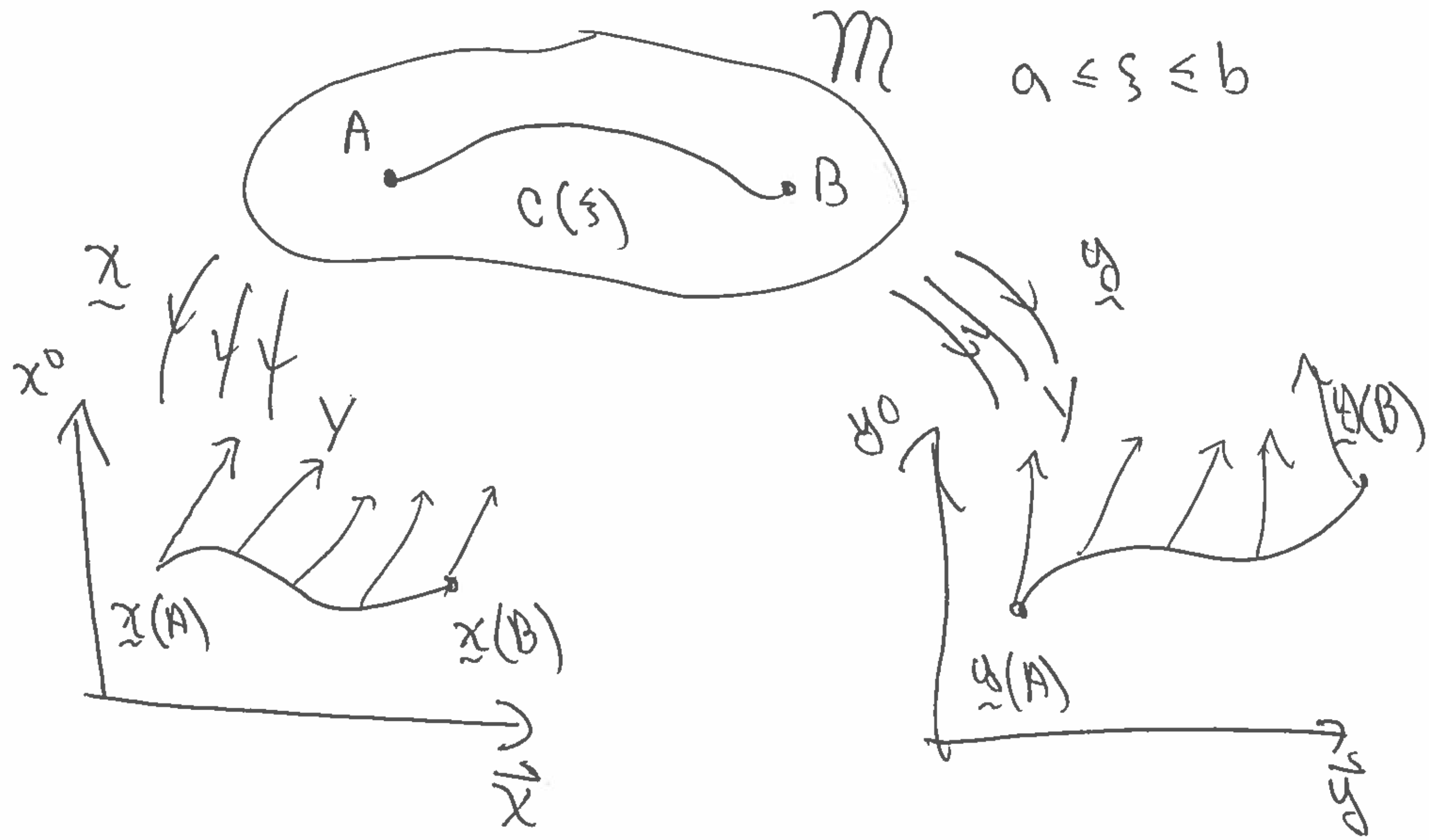
~~Note~~: All such coordinate systems are obtained from $\underset{\sim}{x}$ by taking a Lorentz transformation (neglecting 2nd order errors)

$$g_{\alpha\beta} = \eta_{\alpha\beta} = A^i_\alpha g_{ij} A^j_\beta = A^i_\alpha \eta_{ij} A^j_\beta + O(|\underset{\sim}{x}|^2) \quad \Bigg| \quad \begin{array}{l} x^i = A^i_\alpha y^\alpha \\ \dfrac{\partial x^i}{\partial y^\alpha} = A^i_\alpha \end{array}$$

- In special relativity – freefall paths are straight lines, and parallel translation of vectors (I.e., how non-rotating vectors aligned with gyroscopes are transported along freefall paths) keeps components constant, so angles with coordinate axes are held constant.

In GR, we want to give a mathematically precise prescription for ||-translation of vectors in <u>any coord system</u>, so that in locally inertial coordinates, the translation agrees with special relativity to within 2nd order errors. Turns out, this is uniquely determined,

• Connection for ∥-translation:



Given curve $c(\xi)$, $a \le \xi \le b$. We want to give a condition on vector field $Y = y^i \frac{\partial}{\partial x^i} = \bar{y}^\alpha \frac{\partial}{\partial y^\alpha}$ such that it be parallel along curve $c(\xi)$ with tangent vector $X = \dot{x}^i \frac{\partial}{\partial x^i} = \dot{y}^\alpha \frac{\partial}{\partial y^\alpha}$

Here: $X = X(\xi)$, $Y = Y(\xi)$

- Because everything is linear in the tangent space, we give the condition on $Y(\xi)$ as a condition on its derivative wrt ξ.

Condition in $\underset{\sim}{x}$-coords:

$$\frac{dy^i}{d\xi} = -\Gamma^i_{jk}\, y^j(\xi)\, \underbrace{\frac{dx^k}{d\xi}}_{\dot{x}^k \frac{\partial}{\partial x^k} = X}$$

"Simplest condition linear in y^j & $\dot{x}^k \equiv X^k$"

<u>Problem</u>: Find $\Gamma^i_{jk}(\underset{\sim}{x})$ for each coord system, such that $\Gamma^i_{jk}(P) = 0$ at the center of locally inertial coordinates where $g_{ij} = \eta_{ij}\big|_P$ & $g_{ij,k}\big|_P = 0$.

That is: If $\Gamma^i_{jk} = 0$ at $\underset{\sim}{x}(P) = \underset{\sim}{x}_0$,

$g_{ij}(\underset{\sim}{x}_0) \overset{=\eta_{ij}}{}$ $g_{ij,k}(\underset{\sim}{x}_0) = 0$, then

$$\frac{dy^i}{d\xi}(\xi) = -\underbrace{\Gamma^i_{jk}(\xi)}\, y^j(\xi)\, X^k(\xi)$$

$$\underset{\sim}{x}(\xi_0) = \underset{\sim}{x}_0$$

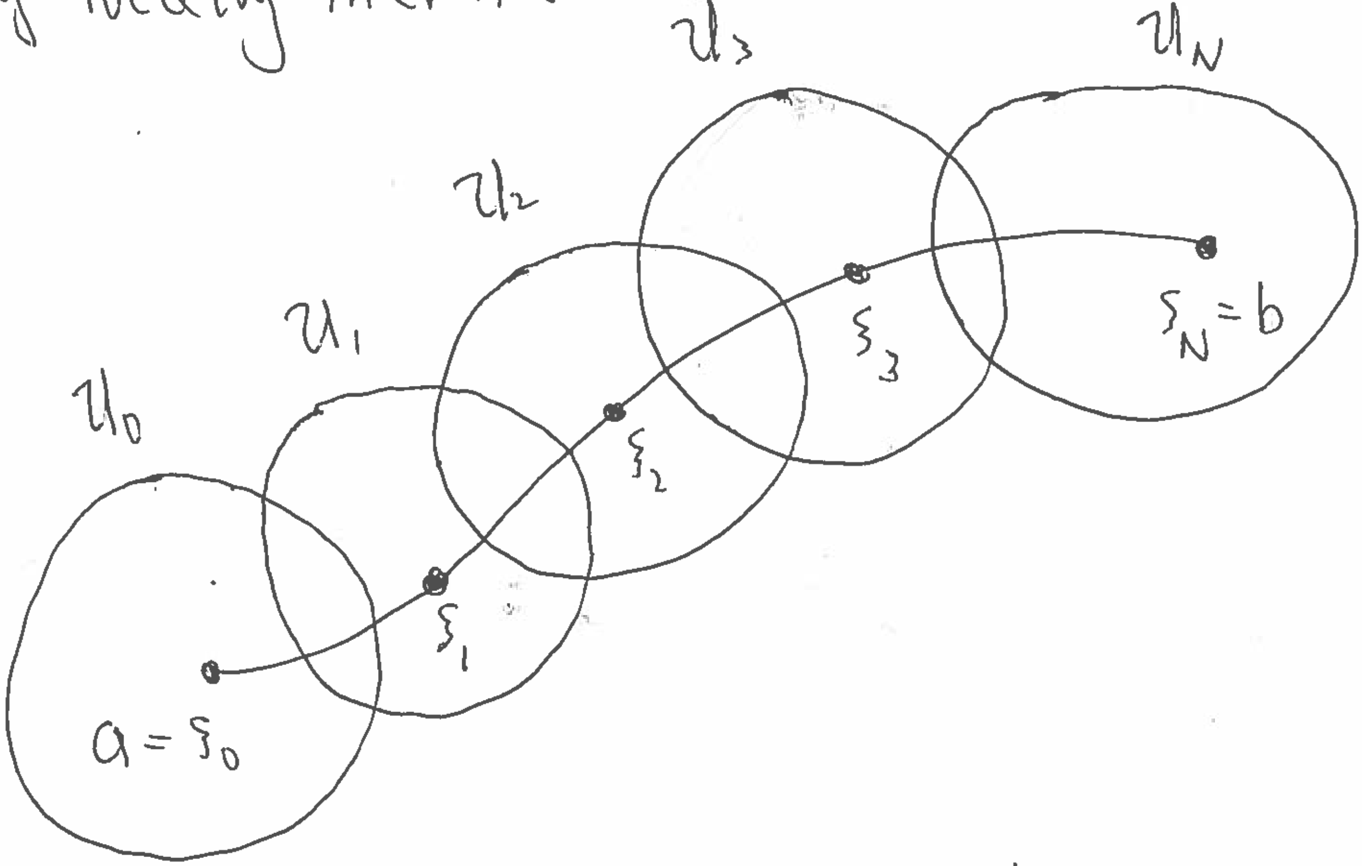
$$O(\xi - \xi_0) \qquad \xi - \xi_0 = \Delta\xi$$

$$\frac{dy^i}{d\xi}(\xi) = O(\xi - \xi_0)$$

$$\frac{y^i(\xi+\Delta\xi) - y^i(\xi)}{\Delta\xi} = O(\Delta\xi)$$

$$\boxed{y^i(\xi+\Delta\xi) = y^i(\xi) + O(|\Delta\xi|^2)}$$

Then our theory reproduces "Λ-translation by locally inertial frames"



I.e. Let $\xi_k = a + k\Delta\xi$, $\Delta\xi = \frac{b-a}{N}$, $N \sim \frac{1}{\Delta\xi}$

$P_k = c(\xi_k)$, let $\mathcal{U}_k$ be a locally inertial frame centered at P_k

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Then: $\cancel{Y^2(\xi_{k+1}) = Y^2(\xi_k) + O(|\Delta\xi|^2)}$

$\cancel{Y^i(b) = Y^i}$ $\quad Y^i(\xi_{k+1}) = Y^i(\xi_k) + O(|\Delta\xi|^2)$

- $\parallel$-translate y^i as constant in each locally inertial coord system.
- Transform y^i from $\mathcal{U}_k$ to $\mathcal{U}_{k+1}$ using the coordinate transformation on the overlap
- The process incurs an error in $\parallel$ translation of order $O(|\Delta\xi|^2)$ in each coordinate system.
- Let Y^i_N be the approx $\parallel$-trans. by locally inertial frames, & $Y^i_{\parallel}$ the exact $\parallel$-translation

$$Y^i_N = Y^i_{\parallel} + \underbrace{\sum_{k=1}^{N} O(|\Delta\xi|^2)}_{\frac{b-a}{\Delta\xi} \text{ errors of order } |\Delta\xi|^2} = Y^i_{\parallel} + O(\Delta\xi)$$

- <u>Conclude</u>: If $\parallel$-translation is given by
$$\frac{dy^i}{d\xi} = -\Gamma^i_{jk}\, y^j x^k \qquad (*)$$
for some coord dependent Γ^i_{jk} which transform so that in $\underset{\sim}{y}$-coords
$$\frac{dy^\alpha}{d\xi} = -\bar{\Gamma}^\alpha_{\beta\gamma}\, y^\beta x^\gamma$$
(making the condition <u>covariant</u>)

Such That: $\Gamma^i_{jk}(P) = 0$ in locally inertial coords at P,

<u>Then</u>

(*) Reproduces "$\parallel$-translation by locally inertial frames".

Theorem ①: Γ^i_{jk} transforms so as to make condition (*) covariant iff

$$\boxed{\bar{\Gamma}^{\alpha}_{\beta\gamma} = \underbrace{\Gamma^i_{jk} \frac{\partial x^j}{\partial y^\beta}\frac{\partial x^k}{\partial y^\gamma}\frac{\partial y^\alpha}{\partial x^i}}_{\text{tensorial}} + \underbrace{\frac{\partial^2 x^\ell}{\partial y^\beta \partial y^\gamma}\frac{\partial y^\alpha}{\partial x^\ell}}_{\text{non-tensorial}}} \quad (\Gamma)$$

Γ is the one fundamental object in differential geometry that does not transform like a tensor

<u>Theorem</u> 2 $\Gamma^i_{jk}(P) = 0$ when $g_{ij}(P) = \eta_{ij}$ and $g_{ij,k}(P) = 0$ iff

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\sigma} \{ -g_{jk,\sigma} + g_{\sigma j,k} + g_{k\sigma,j} \}$$

<u>Defn</u>: $c(\xi)$ is a geodesic (freefall path) if its tangent vector

$$\mathbf{X} = X^i \frac{\partial}{\partial x^i} = \dot{x}^i \frac{\partial}{\partial x^i}$$

satifies

$$\frac{d^2 x^i}{d\xi^2} = -\Gamma^i_{jk} X^j X^k$$

all along the curve. I.e., the tangent vector to $c(\xi)$ is parallel along $c(\xi)$.

"**Cor**": The condition that $c(\xi)$ be a geodesic is just the condition that $c(\xi)$ be a straight line in each locally inertial frame (to within errors $\mathcal{O}(|\underset{\sim}{x} - \underset{\sim}{x}(P)|^2)$).

"pf"

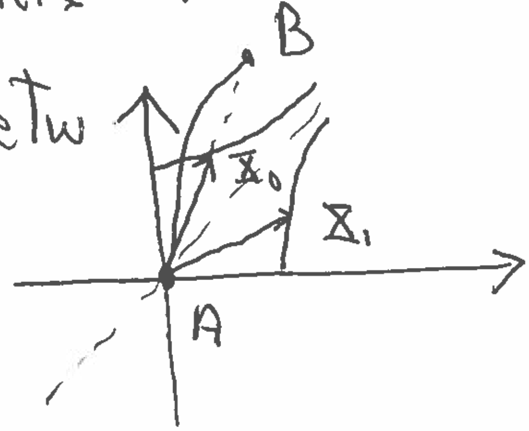
$$\frac{dX^i}{d\xi} = \underbrace{-\Gamma^i_{jk}(\xi)}_{\substack{\equiv 0 \text{ at } \xi = \xi_0 \\ \text{if } g_{ij}(\xi_0) = g_{ij,k}(\xi_0) = 0 \\ \Rightarrow O(\xi - \xi_0) = O(\Delta\xi)}} X^j X^k$$

$$\therefore X^i(\xi_0 + \Delta\xi) = X^i(\xi) + O(|\Delta\xi|^2)$$

$\Rightarrow \underset{\sim}{x}(\xi)$ is a straight line to within errors $O(\Delta\xi^2) = O(|\underset{\sim}{x} - \underset{\sim}{x}_0|^2)$

- Geodesics as critical points of length:

Recall that in special relativity, moving observers "age slower". This tells us that the straight line in a locally inertial frame maximizes proper time among all nearby curves that go betw same points – ie



The proper time change betw A & B is the aging = elapsed time for observer fixed in frame with $\vec{AB}$ as its timelike direction. Other curves that take $A \to B$ have clocks that are slower $\Rightarrow$ straight line maximized arclength on timelike curves. Since geodesics are "locally straight" $\Rightarrow$ they should maximize proper time (locally)

Defn: Geodesics are critical points of the action – Ie, for timelike curve $c(\xi)$,

Define

$$A[c(\cdot)] = \int_A^B \left\| \frac{dc}{d\xi} \right\|^2 d\xi \qquad \begin{array}{l} c(a) = A \\ c(b) = B \end{array}$$

Let $\eta(\xi)$ satisfy $\eta(a) = 0 = \eta(B)$, so $c(\xi) + \varepsilon\eta(\xi)$ is any nearby curve taking $A \to B$. Then

$$\frac{d}{d\varepsilon} \int_A^B \left\| \frac{d(c+\varepsilon\eta)}{d\xi} \right\|^2 d\xi = 0$$

reproduces

$$\frac{d^2x^i}{ds^2} = \Gamma^i_{jk} \dot{x}^i \dot{x}^j$$

for metric connections. Pf Euler-Lagrange

(See Dubrovin, Fomenko, Novikov Modern Geometry – Vol I Chapter 5)

Proof of Theorem 1

Let $X = x^i \frac{\partial}{\partial x^i} = \bar{X}^\alpha \frac{\partial}{\partial y^\alpha}$, $Y = y^i \frac{\partial}{\partial x^i} = \bar{Y}^\alpha \frac{\partial}{\partial y^\alpha}$

be vector fields. Then assuming

$$\frac{d}{d\xi} y^i(\xi) = -\Gamma^i_{jk}\, y^j x^k, \qquad (*)$$

we find $\frac{d}{d\xi} \bar{y}^\alpha(\xi)$. Then

$$\underbrace{\frac{d}{d\xi}\left(\bar{y}^\alpha \frac{\partial x^i}{\partial y^\alpha}\right)} = -\Gamma^i_{jk} \frac{\partial x^j}{\partial y^\beta} \frac{\partial x^k}{\partial y^\gamma} \bar{y}^\beta \bar{X}^\gamma$$

$$\left(\frac{d}{d\xi} \bar{y}^\alpha(\xi)\right) \frac{\partial x^i}{\partial y^\alpha} + \underbrace{\bar{y}^\alpha \frac{d}{d\xi} \frac{\partial x^i}{\partial y^\alpha}}_{\bar{y}^\gamma \bar{X}^\beta \frac{\partial^2 x^i}{\partial y^\beta \partial y^\gamma}}$$

change summation to γ

$$\frac{d}{d\xi} \frac{\partial x^i}{\partial y^\alpha}(\underset{\sim}{y}(\xi)) = \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} \underbrace{\frac{dy^\beta}{d\xi}}_{\bar{X}^\beta}$$

Thus —

$$\left(\frac{d}{d\xi}\,\bar{y}^{\alpha}(\xi)\right)\frac{\partial x^i}{\partial y^\alpha} = -\Gamma^i_{jk}\,\frac{\partial x^j}{\partial y^\beta}\,\frac{\partial x^k}{\partial y^\gamma}\,\bar{y}^\beta\,\bar{X}^\gamma$$

$$-\frac{\partial^2 x^i}{\partial y^\beta \partial y^\gamma}\,\bar{y}^\beta\,\bar{X}^\gamma$$

Multiplying thru by $\left(\frac{\partial x^i}{\partial y^\alpha}\right)^{-1} \equiv \left(\frac{\partial y^\alpha}{\partial x^i}\right)$ and factoring out $\bar{y}^\beta$ & $\bar{X}^\gamma$ gives

$$\frac{d}{d\xi}\,\bar{y}^\alpha(\xi) = -\left\{\Gamma^i_{jk}\,\frac{\partial x^j}{\partial y^\beta}\frac{\partial x^k}{\partial y^\gamma}\frac{\partial y^\alpha}{\partial x^i} + \frac{\partial y^\alpha}{\partial x^i}\frac{\partial^2 x^i}{\partial y^\beta \partial y^\gamma}\right\}$$

$$\cdot\,\bar{y}^\beta\cdot\bar{y}^\gamma$$

$$\{\} = \bar{\Gamma}^\alpha_{\beta\gamma}\quad \checkmark$$

Proof of Theorem 2 Assume (Γ) holds and $\Gamma^i_{jk}(p) = 0$ if $g_{ij}(p) = \eta_{ij}$, $g_{ij,k}(p) = 0$ in x-coords. Then in any other coord system y, assuming (Γ), we have

$$\bar{\Gamma}^{\gamma}_{\alpha\beta}(p) = \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} \frac{\partial y^\gamma}{\partial x^i}. \qquad \left(\begin{array}{l}\text{we only need}\\ \text{formulas @ } p\end{array}\right)$$

Now in y-coords,

$$\bar{g}_{\alpha\beta} = \frac{\partial x^i}{\partial y^\alpha} \eta_{ij} \frac{\partial x^j}{\partial y^\beta} \qquad (g)$$

Differentiating (and using $g_{ij,k} = 0$ at p)

$$\bar{g}_{\alpha\beta,\gamma}(p) = \underbrace{\eta_{ij} \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\gamma} \frac{\partial x^j}{\partial y^\beta}}_{\Delta_{\alpha\gamma,\beta}} + \underbrace{\eta_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial^2 x^j}{\partial y^\beta \partial y^\gamma}}_{\Delta_{\beta\gamma,\alpha}}$$

Equality of mixed partials gives

$$\Delta_{\alpha\gamma,\beta} = \Delta_{\gamma\alpha,\beta}.$$

Now compute

$$-g_{\alpha\beta,\gamma} + g_{\gamma\alpha,\beta} + g_{\beta\gamma,\alpha} = -\cancel{\Delta_{\alpha\gamma,\beta}} - \cancel{\Delta_{\beta\gamma,\alpha}} + \cancel{\Delta_{\gamma\beta,\alpha}} + \Delta_{\alpha\beta,\gamma} + \Delta_{\beta\alpha,\gamma} + \cancel{\Delta_{\gamma\alpha,\beta}}$$

$$= 2\Delta_{\alpha\beta,\gamma} = 2\eta_{ij} \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} \frac{\partial x^j}{\partial y^\gamma}$$

$$\therefore \frac{1}{2} g^{\gamma\sigma} \left\{ -g_{\alpha\beta,\sigma} + g_{\sigma\alpha,\beta} + g_{\beta\sigma,\alpha} \right\}$$

$$= g^{\gamma\sigma} \eta_{ij} \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} \frac{\partial x^j}{\partial y^\sigma} \qquad g^{\gamma\sigma} = \frac{\partial y^\gamma}{\partial x^k} \eta^{kl} \frac{\partial y^\sigma}{\partial x^l}$$

$$= \frac{\partial y^\gamma}{\partial x^k} \eta^{kl} \frac{\partial y^\sigma}{\partial x^l} \eta_{ij} \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} \frac{\partial x^j}{\partial y^\sigma} = \frac{\partial y^\gamma}{\partial x^k} \underbrace{\eta^{kj} \eta_{ij}}_{\delta^k_i \quad k = i} \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta}$$

$$\delta^j_l \qquad l = j$$

Conclude:

$$\frac{1}{2} g^{\gamma\sigma}\left\{ -g_{\alpha\beta,\sigma} + g_{\sigma\alpha,\beta} + g_{\beta\sigma,\alpha} \right\}$$

$$= \frac{\partial y^\gamma}{\partial x^i} \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} = \Gamma^\gamma_{\alpha\beta} \quad \text{as claimed} \checkmark$$

☒ Theory of ODE's —

Basic Existence & Uniqueness ($\exists !$) Thm:

(ODE) $\quad \dot{\underset{\sim}{y}} = f(t, \underset{\sim}{y}) \qquad \underset{\sim}{y} = (y^1, \dots, y^n)^{tr}$

$\qquad\qquad\qquad\qquad\qquad\qquad f = (f_1, \dots, f_n)^{tr}$

$$\begin{pmatrix} y^1 \\ \vdots \\ y_n \end{pmatrix}' = \begin{pmatrix} f_1(t, \underset{\sim}{y}) \\ \vdots \\ f_n(t, \underset{\sim}{y}) \end{pmatrix} \qquad \dot{\underset{\sim}{y}} \equiv \frac{d}{dt} \underset{\sim}{y}(t) = \underset{\sim}{y}'(t)$$

(Initial Condition) $\quad \underset{\sim}{y}(t_0) = \underset{\sim}{y}_0$ given.

Initial Value Problem: Find a solution $\underset{\sim}{}(t)$ that solves

$$\begin{aligned} \dot{\underset{\sim}{y}}(t) &= f(t, \underset{\sim}{y}(t)) \\ \underset{\sim}{y}(t_0) &= \underset{\sim}{y}_0 \end{aligned} \qquad (*)$$

in some nbhd $|t - t_0| < \varepsilon$, some $\varepsilon > 0$.

For nonlinear problems, solutions only exist <u>locally</u> - For linear problems they exist <u>globally</u>

Thm ① (local, nonlinear) Assume f is continuous and Lipschitz continuous in $\underset{\sim}{y}$. Then $\exists\ \varepsilon > 0$ st $\exists!$ solution $\underset{\sim}{y}(t)$ of (*); i.e., $\exists$ functions $\underset{\sim}{y}(t) = (y_1(t), \ldots, y_n(t))$ defined for $|t-t_0| < \varepsilon$ such that

$$\underset{\sim}{y}'(t) = f(t, \underset{\sim}{y}(t))$$
$$\underset{\sim}{y}(t_0) = \underset{\sim}{y}_0$$

Defn: f Lipschitz continuous if $\exists$ const C such that

$$|f(t, \underset{\sim}{y}_2) - f(t, \underset{\sim}{y}_1)| \leq C\, |\underset{\sim}{y}_2 - \underset{\sim}{y}_1|$$

$\forall\ \underset{\sim}{y}_1, \underset{\sim}{y}_2$.

Thm 2: (global, linear) If f is linear, so

$$\dot{\underset{\sim}{y}} = f(t, \underset{\sim}{y}) = \underset{n\times n}{A(t)}\, \underset{\sim}{y} + \underset{n\times 1}{\underset{\sim}{b}(t)},$$

then ! solution exists on any interval $[a,b]$ where A & $\underset{\sim}{b}$ are continuous.

Ex 1 Assume you are given a curve $c(\xi)$, so in $\underset{\sim}{x}$-coordinates

$$\underset{\sim}{x}(\xi) = \underset{\sim}{x} \circ c\,(\xi) = (x^1(\xi), \ldots, x^n(\xi))$$

Given $Y(c(a)) = Y_0^i \frac{\partial}{\partial x^i}\Big|_{c(a)}$, we wish to Γ-translate Y from $c(a)$ to $c(b)$.

So we solve for $Y^i(\xi)$:

$$\frac{dY^i}{d\xi} = -\Gamma^i_{jk}\, \dot{x}^j\, Y^k \qquad (\Gamma)$$

$$Y^i(a) = Y^i_0$$

④

Since $\Gamma^i_{jk} \equiv \Gamma^i_{jk}(\underset{\sim}{x}(\xi))$ are known smooth functions, and $\dot{x}^j(\xi)$ is known, this is a <u>linear</u> equation of form

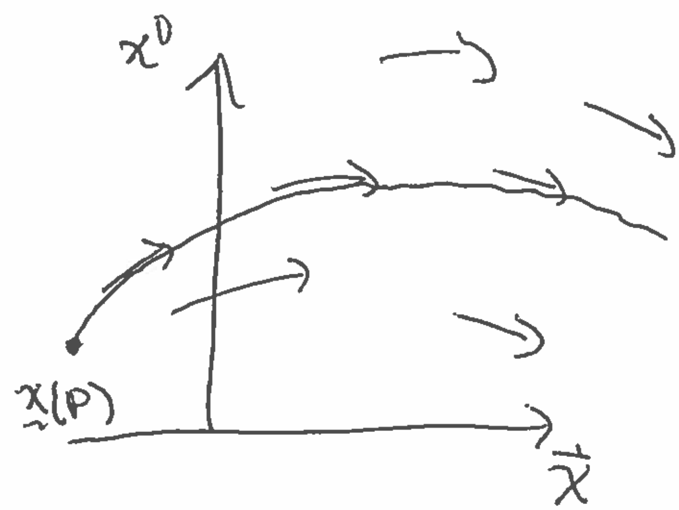
$$\frac{dy^i}{d\xi} = \underset{4\times4}{A^i_k(\xi)}\ \underset{4\times1}{y^k}$$

so with $\xi \equiv t$, $a = t_0$, Thm B says $\exists!$ solution $y^i(\xi)$ ✓

Ex (2) Assume we have a vector field $\underline{X}$ defined on spacetime $\mathcal{M}$, so in each coord system it take form

$$\underline{X} = X^i \frac{\partial}{\partial x^i} \quad \text{(defined at all } \underset{\sim}{x}\text{)}$$

So $X^i \equiv X^i(\underset{\sim}{x})$ are smooth functions of $\underset{\sim}{x}$. We wish to find a curve



$\underset{\sim}{x}(\xi)$, starting at $\underset{\sim}{x}(\xi_0) = \underset{\sim}{x}(P) = \underset{\sim}{x}_0$, such that its tangent vector is $\underline{X}$ at each point on the curve.

Thus we need to solve:

$$\underline{X} = \dot{x}^i \frac{\partial}{\partial x^i} \quad \text{along some curve } \underset{\sim}{x}(\xi).$$

That is: We need to solve:

$$\dot{\underset{\sim}{x}}^i = X^i$$

$$\underset{\sim}{x}(0) = \underset{\sim}{x}(P) \equiv \underset{\sim}{x}_0$$

Ie we need a function (curve) $\underset{\sim}{x}(\xi)$ st

$$\underbrace{\dot{\underset{\sim}{x}}^i(\xi)}_{\substack{\text{component}\\ \text{of tangent}\\ \text{vector to curve}}} = \underbrace{X^i(\underset{\sim}{x}(\xi))}_{\substack{\text{given component @ each}\\ \underset{\sim}{x}\text{, given by vector field } \underline{X}}}$$

$$\underset{\sim}{x}(0) = \underset{\sim}{x}_0$$

where the curve starts.

$\Rightarrow$ Nonlinear ODE with $f^i(\underset{\sim}{x}) = X^i(\underset{\sim}{x})$.

<u>Conclude</u>: Thm ① applies if $X^i(\underset{\sim}{x})$ are Lipschitz continuous $\Rightarrow \exists!$ solution $\underset{\sim}{x}(\xi)$ locally, for $|\xi| < \varepsilon$, some $\varepsilon > 0$.

<u>Defn</u>: $\underset{\sim}{x}(\xi)$ is called the integral curve of $\underline{X}$ thru P if $\underset{\sim}{x}(\xi_0) = P$ some ξ_0.

<u>Note</u>: If $f : \mathcal{M} \to \mathbb{R}$ scalar function with representation $f \circ \underset{\sim}{x}^{-1} \equiv f(\underset{\sim}{x})$, we have that $\underline{X} = X^i \frac{\partial}{\partial x^i}$ can be viewed as a differential operator on f, i.e. $\underline{X}(f) = X^i \frac{\partial}{\partial x^i} f = \nabla_{\underline{X}} f$.

Alternatively, if $\underset{\sim}{x}(\xi)$ is integral curve of $\underline{X}$,

$$\frac{d}{d\xi} f(\underset{\sim}{x}(\xi)) = \frac{\partial f}{\partial x^i} \frac{dx^i}{d\xi} = \dot{x}^i \frac{\partial}{\partial x^i} f = X^i \frac{\partial}{\partial x^i} f = \underline{X}(f)$$

"$\nabla_{\underline{X}} f$" is derivative of f along integral curve of $\underline{X}$!

<u>Defn</u>: If f is indept of t, so

$$\dot{\underset{\sim}{y}} = f(\underset{\sim}{y}) \qquad (f(\cancel{t}, \underset{\sim}{y}) = f(\underset{\sim}{y}))$$

so f depends on t only thru unknown function $\underset{\sim}{y}(t)$, then we say ODE is <u>autonomous</u>

<u>Cor</u>: Solution curves of an autonomous ~~system~~ ODE never intersect, and "foliate" any nbhd $U_{\underset{\sim}{y}}$.

<u>Application</u>: Integral curves of vector fields define a unique curve thru every point, all non-intersecting, and we can use this to define a natural coordinate ξ that goes with any vector field $\underset{\sim}{X}$. I.e., if $x^i = \xi$, then $\frac{\partial}{\partial x^i} = \underset{\sim}{X}$ makes $\underset{\sim}{X}$ a coordinate vector field.

Ex: 2x2 autonomous systems are <u>special</u> because the entire family of solution curves is determined by the structure of the singular points.

<u>Application</u>:

$$Z' = -3W\left(\tfrac{4}{3} + Z\right)$$

$$W' = -\tfrac{1}{6}Z - \tfrac{1}{3}W - W^2$$

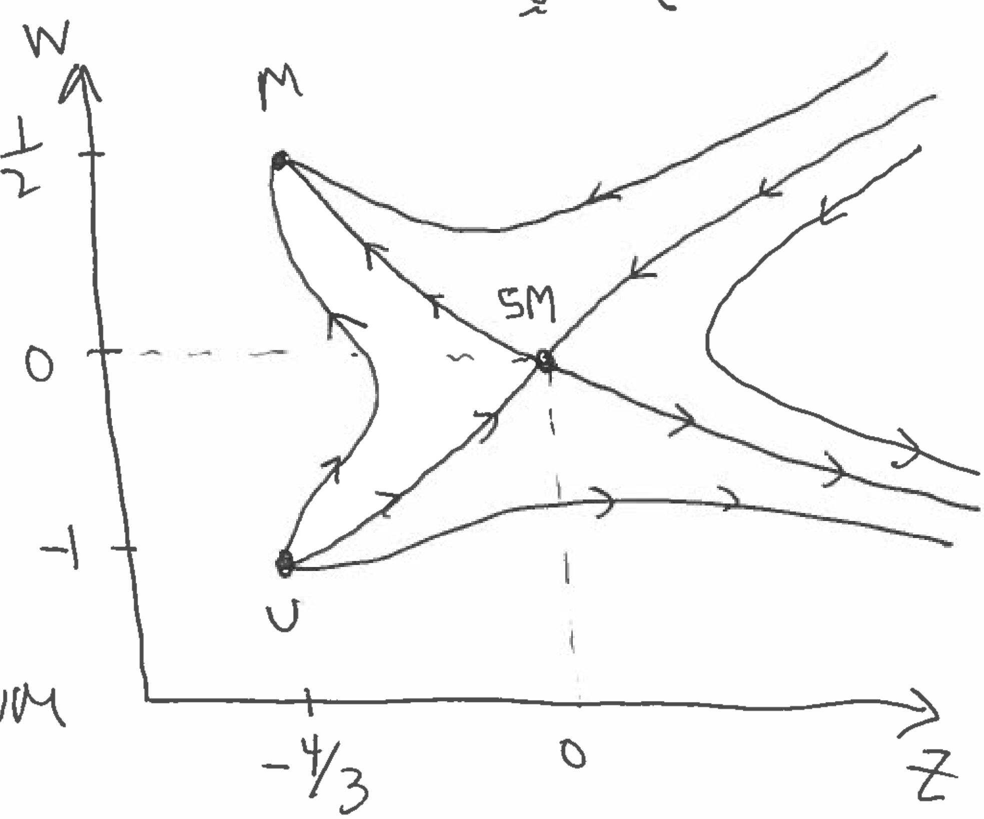
$$\Leftrightarrow \begin{pmatrix} Z(t) \\ W(t) \end{pmatrix}' = f\begin{pmatrix} Z(t) \\ W(t) \end{pmatrix} \Leftrightarrow \underset{\sim}{y}' = f(\underset{\sim}{y}), \quad \underset{\sim}{y} = \begin{pmatrix} z \\ w \end{pmatrix}$$

Solution:

M - stable

U - unstable

SM - unstable Saddle ~ unstable pendulum



- **Defn**: We say Γ's define a connection

$\Gamma^i_{jk} \equiv$ Christoffel Symbols (of 2nd kind)

Cor ① The difference betw two connections is a tensor. ✓ ("Correction cancels out")

Cor ② Given Γ^i_{jk}, $\Gamma^i_{jk} - \Gamma^i_{kj}$ transforms like a tensor ✓ ("Correction cancels out")

Defn: $\Gamma^i_{jk} - \Gamma^i_{kj} \equiv T^i_{jk}$ is the torsion tensor. (Measures "twist rel to nearby geodesics")

Cor ③ Symmetry $\equiv$ "$\Gamma^i_{jk} = \Gamma^i_{kj}$" is a coord indept prop of connections

Pf. $T^i_{\cdot\cdot} = 0$ in one coord syst $\Rightarrow 0$ in all. ✓

Cor ④ $\Gamma^{i}_{jk}{}^{jk} = 0$ @ P in $\underset{\sim}{x}$-coords $\Rightarrow$ Γ symmetric ($T^i_{jk} = 0$!)

17 + 18 + 19

@ <u>Covariant</u> <u>Derivative</u>: $\nabla_{\underline{X}} \underline{Y}$ defined by Γ

- Given 2 vector fields $\underline{X}, \underline{Y}$
- Let $\underset{\sim}{x}(\xi)$ be integral curve of $\underline{X}$ starting @ $\underset{\sim}{x}(0) = p$, so $\frac{d\underset{\sim}{x}(\xi)}{d\xi} = \underline{X}_{\underset{\sim}{x}(\xi)}$
- Let $\underline{Y}(\xi) = \underline{Y}_{\underset{\sim}{x}(\xi)}$

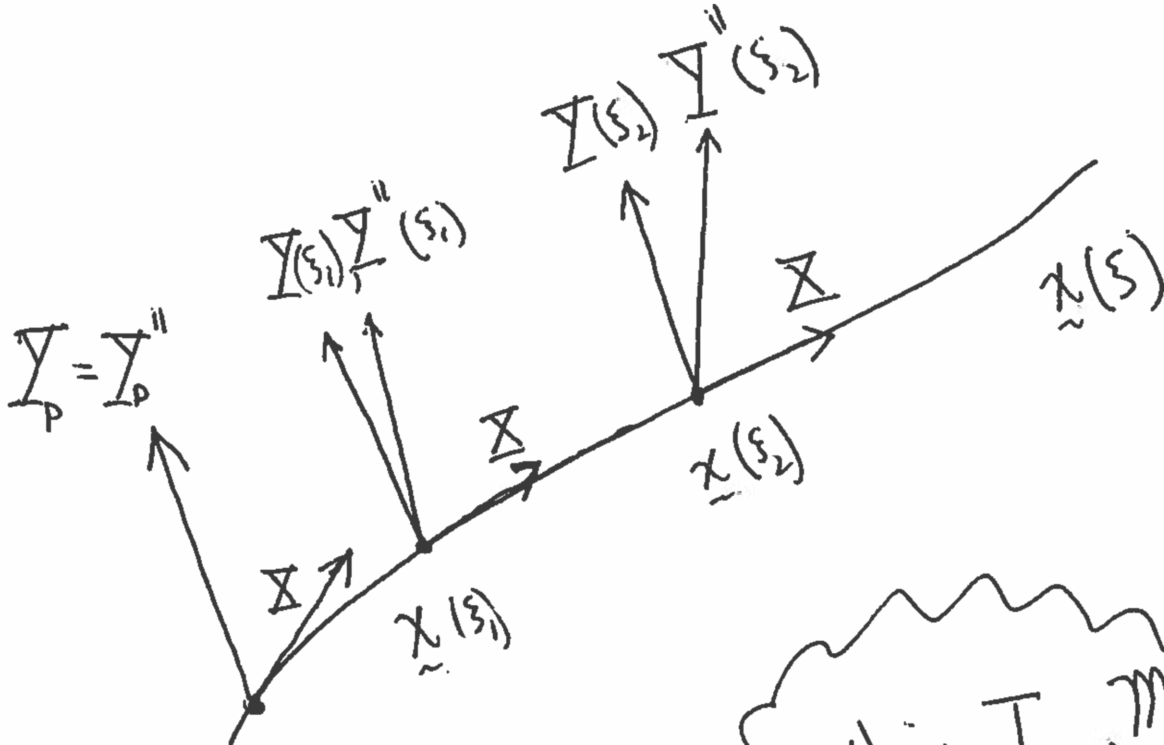
$$\underline{Y}_{\parallel}(\xi) \equiv \parallel\text{-trans of } \underline{Y}_p \text{ to } \underset{\sim}{x}(\xi) \text{ along } C_X$$

- <u>Defn</u>: $\nabla_X \underline{Y}\Big|_p = \lim_{\xi \to 0} \frac{\underline{Y}(\xi) - \underline{Y}_{\parallel}(\xi)}{\xi}$

Since both $\underline{Y}(\xi)$ & $\underline{Y}_{\parallel}(\xi)$ are coord indept, this gives coord indept notion of deriv of vector field $\underline{Y}$ in $\underline{X}$ direction.

(only depends on $\underline{X}$ @ $\underline{X}(p)$!)

Picture $\nabla_X \underset{\sim}{Y}$:



$\underset{\sim}{x}(0) = p$

both in $T_{\underset{\sim}{x}(\xi)}\mathcal{M}$

$$\left(\nabla_{\underset{\sim}{X}} \underset{\sim}{Y}\right)_p = \lim_{\xi \to 0} \frac{\underset{\sim}{Y}(\xi) - \underset{\sim}{Y}_{\parallel}(\xi)}{\xi}$$

- The covariant derivative corrects vector differentiation to a tensor operation: I.e.,

$$\left(\nabla_{\underline{X}}\underline{Y}\big|_p\right)^i = \lim_{\xi\to 0}\frac{Y^i(\xi)-Y^i(0)}{\xi} + \lim_{\xi\to 0}\frac{Y^i(0)-Y^i_{\parallel}(\xi)}{\xi}$$

i-component in x-coords

We only have a coord way to express this limit!

$$= \underline{X}(\underline{Y})^i - \frac{dY^i_{\parallel}}{d\xi}$$

$$= \underline{X}(\underline{Y})^i + \Gamma^i_{jk}\, Y^j X^k$$

coord dept, not a tensor

Γ gives us a coord expression for a coord indept thing

$$\underline{X}(\underline{Y})^i = \frac{d}{d\xi}\, Y^i(\xi)$$

- In coordinates:

$$(\nabla_{\underline{X}}\underline{Y})^i = \frac{dY^i}{d\xi} - \frac{dY^i_{\parallel}}{d\xi} = \underbrace{X^\sigma Y^i_{,\sigma}}_{\text{" }\underline{X}(Y^i(\underset{\sim}{x}))\ =\ \frac{d}{d\xi}Y^i(\xi)\text{ "}} + \underbrace{\Gamma^i_{jk}\, Y^j X^k}_{\text{corrects } X(Y) \text{ to a tensor}}$$

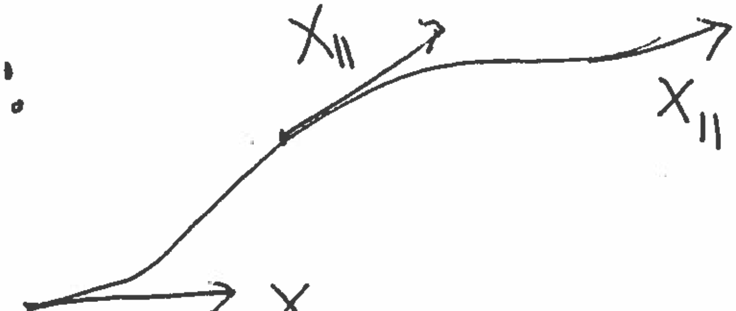
- Conclude: ∇ gives a coord indep't expression to the Γ's

- Defn: $\underline{Y}$ is parallel along $c(\xi)$ if

$$\nabla_{\underline{X}} \underline{Y} = 0 \,, \quad \underline{X} = \frac{dc}{d\xi}$$

- Defn: a curve $\gamma(\xi)$ is a geodesic of Γ if $\underline{X} = \frac{d\gamma}{d\xi}$ is parallel along γ.

Geodesic Equation:



$\nabla_{\underline{X}} \underline{X} = 0 \iff$

$$(\nabla_{\underline{X}} \underline{X})^i = X^j \frac{\partial}{\partial x^j} X^i_{\gamma(\xi)} + \Gamma^i_{jk}(\gamma(\xi)) X^j X^k = 0$$

Since $\dot{\gamma}^i(\xi) = X^i \Rightarrow X^j \frac{\partial}{\partial x^j} \dot{\gamma}^i(\xi) = \ddot{\gamma}^i(\xi)$

$$\iff \boxed{\ddot{\gamma}^i + \Gamma^i_{jk} \dot{\gamma}^j \dot{\gamma}^k = 0}$$

• <u>Conclude</u>: ∇ gives a coordinate indept expression to the Γ's

<u>Properties</u>: ($X_i \in T_pM$, Y a <u>vector field</u>)

① $\nabla_{aX_1+bX_2} Y = a\nabla_{X_1} Y + b\nabla_{X_2} Y$ $\left\{\begin{array}{l}\text{any smooth fn's} \\ a, b : M \to \mathbb{R}\end{array}\right.$

② $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$

③ $\nabla_X [f(p) Y] = f(p)\nabla_X Y + \underbrace{X(f)}\nabla_X Y$

define $\nabla_X f = X(f)$ so Liebniz rule holds

④ $\nabla_X Y - \nabla_Y X = [X, Y] \equiv L_X Y$ (when $\Gamma^i_{jk} = \Gamma^i_{kj}$)

Pf ④: $\nabla_X Y - \nabla_Y X = X(Y) + \Gamma^i_{jk} y^j x^k$

$$- Y(X) - \Gamma^i_{jk} x^j y^k$$

$$= \overbrace{X(Y) - Y(X)}^{[X,Y]} + T^i_{jk} X^j Y^k$$

We assume Γ symmetric here on

◈ Extend ∇ to covectors ω by requiring:

$$(\nabla_{\underset{\sim}{X}}\omega)(\underset{\sim}{Y}) = \nabla_{\underset{\sim}{X}}(\omega(\underset{\sim}{Y})) \quad \forall\ \underset{\sim}{Y} \text{ st } \nabla_{\underset{\sim}{X}}\underset{\sim}{Y} = 0$$

"so that $\nabla_{\underset{\sim}{X}}\omega = 0$ when $\omega(\underset{\sim}{Y})$ evaluates parallel vector fields $\underset{\sim}{Y}$ along $\underset{\sim}{x}(s)$ as constant."

<u>That is</u>: $\nabla_{\underset{\sim}{X}}\omega = (\nabla_{\underset{\sim}{X}}\omega)_{\dot\sigma}\, dx^{\dot\sigma}$

so $(\nabla_{\underset{\sim}{X}}\omega)(\underset{\sim}{Y}) = (\nabla_{\underset{\sim}{X}}\omega)_{\dot\sigma}\, \underset{\sim}{Y}^{\dot\sigma}$

subject to $(\nabla_{\underset{\sim}{X}}\omega)_{\sigma}\, \underset{\sim}{Y}^{\sigma} = \nabla_{\underset{\sim}{X}}(\omega(\underset{\sim}{Y}))$ when $\nabla_{\underset{\sim}{X}}\underset{\sim}{Y} = 0$.

So assume $\nabla_X Y = 0$, and calculate

$$\nabla_X \,\omega(Y) = X(\omega(Y)) = X^i \frac{\partial}{\partial x^i}(\omega_\sigma Y^\sigma)$$

$$= X^i\left(\frac{\partial}{\partial x^i}\omega_\sigma\right)Y^\sigma + X^i \omega_\sigma \frac{\partial}{\partial x^i}(Y^\sigma)$$

$$= \left(X^i \frac{\partial}{\partial x^i}\omega_\sigma\right)Y^\sigma + \omega_\sigma\left(\underbrace{\cancel{(\nabla_X Y)^\sigma}}_{0} - \Gamma^\sigma_{jk} Y^j X^k\right)$$

$$= \left(X^i\frac{\partial}{\partial x^i}\omega_\sigma - \Gamma^\tau_{\sigma k}\,\omega_\tau X^k\right)Y^\sigma$$

$$= (\nabla_X\omega)_\sigma Y^\sigma$$

$$\Rightarrow (\nabla_X \omega)_\sigma = X^i \omega_{\sigma,i} - \Gamma^\tau_{\sigma k}\,\omega_\tau X^k$$

//

We also write

$$\nabla_i Y = \nabla_{\frac{\partial}{\partial x^i}} Y = \left(Y^\sigma_{,i} + \Gamma^\sigma_{ij} Y^j\right)\frac{\partial}{\partial x^\sigma}$$

$$= Y^\sigma_{;i}\,\frac{\partial}{\partial x^\sigma}$$

$$\nabla_i \omega = \omega_{\sigma;i}\, dx^\sigma = \left(\omega_{\sigma,i} - \Gamma^\tau_{\sigma i}\,\omega_\tau\right)dx^\sigma$$

- we can extend ∇ to arb. tensor fields by asking:

$$\left[\nabla_{\underset{\sim}{X}} T\right](X_1, \ldots, X_k, \omega^1, \ldots, \omega^l)$$

$$= \nabla_X \underbrace{\left[T(X_1, \ldots, X_k, \omega^1, \ldots, \omega^l)\right]}_{\text{scalar}}$$

for all $X_1 \cdots X_k \; \omega^1 \cdots \omega^l \; \Lambda$ along $\underset{\sim}{x}(s)$.

<u>Formula</u>:

$\nabla_{\underset{\sim}{X}}\left(T^{\;i}_{j}\, dx^j \otimes \frac{\partial}{\partial x^i}\right)$ has components

$$X^k \frac{\partial}{\partial x^k} T^{\;i}_{j} + \Gamma^{i}_{\sigma\tau} T^{\;\sigma}_{j} X^\tau - \Gamma^{\sigma}_{j\tau} T^{\;i}_{\sigma} X^\tau = (\nabla_X T)^{\;i}_{j}$$

a term for every contravariant indice

a term for every covariant indice.

(*)

Defn: we let ∇T denote the (tensor) with components $T^{j_1 \cdots j_l}_{i_1 \cdots i_k ; j}$ when T has components $T^{j_1 \cdots j_l}_{i_1 \cdots i_k}$.

$$(\nabla X)^i_j = X^i{}_{;j} = X^i{}_{,j} + \Gamma^i_{\sigma j} X^\sigma \quad \text{etc.}$$

Properties:

① $\nabla_X T$ is a tensor for any tensor T.

② $\nabla_X (A \otimes B) = \nabla_X A \otimes B + A \otimes \nabla_X B$

~~[illegible]~~

③ $\nabla_X (T^i{}_i) = (\nabla_X T)^i{}_i$

Moregenerally, ∇_X commutes with contraction.

Ref MTW pg 223

<u>Claim</u>: $\underset{\sim}{X}(\underset{\sim}{Y})$ is not a vector field or tensor. Its defined by

In $\underset{\sim}{x}$-coords: $\underset{\sim}{X}(\underset{\sim}{Y})^i = X^j \frac{\partial}{\partial x^j} Y^i = \frac{d}{ds} Y^i(\underbrace{\underset{\sim}{x}(s)}_{\text{int curve of } \underset{\sim}{X}})$