(II) Proof of Cauchy-Riemann (CR)

☒ <u>Theorem</u> (CR) Assume $f(z) = u(x,y) + iv(x,y)$, and assume u and v are differentiable (in the sense of Real Valued functions $u: \mathbb{R}^2 \to \mathbb{R}$, $v: \mathbb{R}^2 \to \mathbb{R}$) at the <u>point</u> $z_0 = x_0 + iy_0$. Then

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \qquad (D)$$

exists independent of how $\Delta z \to 0$ iff CR holds:

$$\begin{aligned} u_x(x_0, y_0) &= v_y(x_0, y_0) \\ u_y(x_0, y_0) &= -v_x(x_0, y_0). \end{aligned} \qquad (CR)$$

<u>Note</u>: We don't need the derivatives to be continuous, or even defined away from z_0 !

<u>Proof</u> ($\Rightarrow$) For this we must assume $f'(z_0)$ exists in the sense of (D), and prove (CR). This we already did by taking $\Delta z = \Delta x$ and getting $f'(z_0) = u_x + iv_x$; then $\Delta z = i\Delta y$ to get $f'(z_0) = v_y - iu_x \Rightarrow u_x = v_y$ & $u_y = -v_x$ ✓

($\Leftarrow$) This is the hard way. For this we must assume that u and v are differentiable at (x_0, y_0) in the sense of real valued functions together with (CR), and prove that the limit (D) exists independent of how you take $\Delta z \to 0$. ②

- We use the $\mathbb{R}^2$ notation $\underset{\sim}{x} = (x, y)$, $\underset{\sim}{x}_0 = (x_0, y_0)$, $|\underset{\sim}{x}| = \sqrt{x^2 + y^2} = |z|$.

- To start we must recall the definition of differentiability at $\underset{\sim}{x}_0$ for real valued fns:

Defn: $u(x,y) = u(\underset{\sim}{x})$ is differentiable at $\underset{\sim}{x}_0 = (x_0, y_0)$ if there exists a vector $\overrightarrow{(a,b)}$ named $\nabla u(\underset{\sim}{x}_0) = \overrightarrow{(a,b)}$ such that

$$\lim_{\underset{\sim}{x} \to \underset{\sim}{x}_0} \left\{ \frac{u(\underset{\sim}{x}) - u(\underset{\sim}{x}_0)}{|\underset{\sim}{x} - \underset{\sim}{x}_0|} - \overrightarrow{(a,b)} \cdot \frac{\underset{\sim}{x} - \underset{\sim}{x}_0}{|\underset{\sim}{x} - \underset{\sim}{x}_0|} \right\} = 0 \quad \text{(diff)}$$

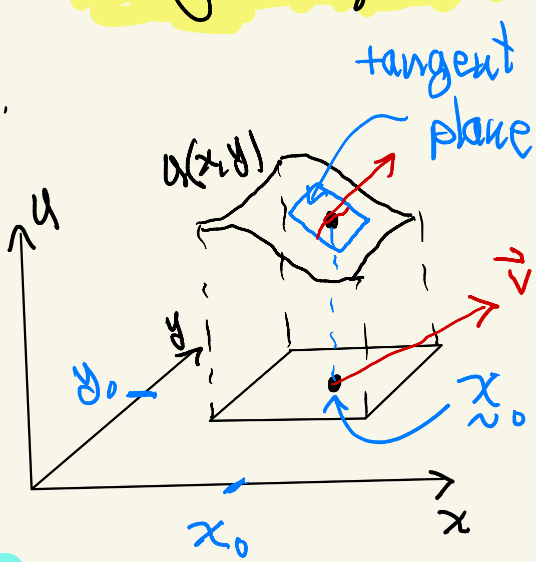
$\overrightarrow{(a,b)} \cdot (\underset{\sim}{x} - \underset{\sim}{x}_0)$ is "linear approximation"

$$\lim_{\underset{\sim}{x} \to \underset{\sim}{x}_0} \left\{ \frac{u(\underset{\sim}{x}) - u(\underset{\sim}{x}_0)}{|\underset{\sim}{x} - \underset{\sim}{x}_0|} - \overrightarrow{(a,b)} \cdot \frac{\underset{\sim}{x} - \underset{\sim}{x}_0}{|\underset{\sim}{x} - \underset{\sim}{x}_0|} \right\} = 0$$

This says: "The difference $u(\underset{\sim}{x}) - u(\underset{\sim}{x}_0)$ agrees with a linear approximation $\overrightarrow{(a,b)} \cdot (\underset{\sim}{x} - \underset{\sim}{x}_0)$ (i.e., a tanget plane) to within an error which tends to zero faster than $|\underset{\sim}{x} - \underset{\sim}{x}_0| = |\Delta z|$ tends to zero"

I.e., $u : \mathbb{R}^2 \to \mathbb{R}$ must have a tangent plane in every direction at $\underset{\sim}{x}_0$.

For example, assuming (diff), if $\underset{\sim}{x} - \underset{\sim}{x}_0 = t\vec{v}$, (i.e., we restrict to the line thru $\underset{\sim}{x}_0$ in direction $\vec{v}$, $\|\vec{v}\| = 1$), gives



$$\frac{d}{dt} u(\underset{\sim}{x}_0 + t\vec{v}) = \overrightarrow{(a,b)} \cdot \vec{v} \equiv \nabla u(\underset{\sim}{x}_0) \cdot \vec{v}$$

(giving the formula for directional derivatives)

Taking $\vec{v} = \vec{e}_1 = \overrightarrow{(1,0)}$, $\vec{e}_2 = \overrightarrow{(0,1)}$ gives

$$\underbrace{\frac{d}{dt} u(\underset{\sim}{x}_0 + t\vec{e}_1)}_{\text{defn of } u_x(\underset{\sim}{x}_0)} = \overrightarrow{(a,b)} \cdot \overrightarrow{(1,0)} = a \Rightarrow u_x = a$$

$$\underbrace{\frac{d}{dt} u(\underset{\sim}{x}_0 + t\vec{e}_2)}_{\text{defn of } u_y(\underset{\sim}{x}_0)} = \overrightarrow{(a,b)} \cdot \overrightarrow{(0,1)} = b \Rightarrow u_y = b$$

Conclude: (diff) implies that $\overrightarrow{(a,b)} = \nabla u(\underset{\sim}{x}_0)$ in the usual sense of partial derivatives.

- Note: the defn of derivative says $u(x,y)$ must have a tangent plane covering every direction and it's not enough to say u_x & u_y exist!

- To make the proof easier, we re-write the limit (diff) using the "order notation"

$$\lim_{\underset{\sim}{x} \to \underset{\sim}{x}_0} \left\{ \frac{u(\underset{\sim}{x}) - u(\underset{\sim}{x}_0)}{|\underset{\sim}{x} - \underset{\sim}{x}_0|} - \overrightarrow{(a,b)} \cdot \frac{\underset{\sim}{x} - \underset{\sim}{x}_0}{|\underset{\sim}{x} - \underset{\sim}{x}_0|} \right\} = 0 \qquad \text{(diff)}$$

Equivalently – multiply thru by $|\underset{\sim}{x} - \underset{\sim}{x}_0|$

$$\underbrace{u(\underset{\sim}{x}) - u(\underset{\sim}{x}_0)}_{\text{change in the output } u} - \underbrace{\overrightarrow{(a,b)}(\underset{\sim}{x} - \underset{\sim}{x}_0)}_{\text{change in } u \text{ along tangent plane}} = \underbrace{o(1)}_{} |\underset{\sim}{x} - \underset{\sim}{x}_0| \qquad \text{(diff)}$$

"litte oh of one" $\equiv$ a function which tends to zero as $\underset{\sim}{x} \to \underset{\sim}{x}_0$

In words: "u agrees with a tangent plane to within an error which tends to zero faster than $\underset{\sim}{x}$ tends to $\underset{\sim}{x}_0$"

- Here $o(1)$ "little oh of one" is a function of $\underset{\sim}{x}$ which tends to zero as $\underset{\sim}{x} \to \underset{\sim}{x}_0$. The idea is that we don't have to keep track of anything about $o(1)$ except $o(1) \xrightarrow[\underset{\sim}{x} \to \underset{\sim}{x}_0]{} 0$

- The order notation lets us replace a limit by an equality!

Conclude: If both $u(x,y)$ and $v(x,y)$ are differentiable at $\underset{\sim}{x}_0 = (x_0, y_0)$, then there exists vectors $\nabla u_0 = \overrightarrow{(a,b)}$ and $\nabla v_0 = \overrightarrow{(c,d)}$ such that

$$u(\underset{\sim}{x}) - u(\underset{\sim}{x}_0) = \overrightarrow{(a,b)} \cdot (\underset{\sim}{x} - \underset{\sim}{x}_0) + o(1)|\underset{\sim}{x} - \underset{\sim}{x}_0|$$

$$v(\underset{\sim}{x}) - v(\underset{\sim}{x}_0) = \overrightarrow{(c,d)} \cdot (\underset{\sim}{x} - \underset{\sim}{x}_0) + o(1)|\underset{\sim}{x} - \underset{\sim}{x}_0|$$

We do not dinstinguish different $o(1)$ terms !

or in vector notation:

$$\begin{pmatrix} u \\ v \end{pmatrix}(\underset{\sim}{x}) - \begin{pmatrix} u \\ v \end{pmatrix}(\underset{\sim}{x}_0) = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{Jacobian} = J(\underset{\sim}{x}_0)} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + o(1)|\underset{\sim}{x} - \underset{\sim}{x}_0|$$

Therefore: If $\begin{pmatrix} u \\ v \end{pmatrix}$ is differentiable at $\underset{\sim}{x}_0$, then we must have

$$J(\underset{\sim}{x}_0) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} - \nabla u(\underset{\sim}{x}_0) - \\ - \nabla v(\underset{\sim}{x}_0) - \end{bmatrix}$$

- Now we need to show $f'(z_0)$ exists at $z = z_0$.

assuming (CR) $u_x = a = d = v_y$, $u_y = b = -c = -v_x$

But $f'(z_0)$ exists if there exists $(\alpha, \beta) \in \mathbb{R}^2$ st

$$\lim_{z \to z_0} \frac{f(z) + f(z_0)}{z - z_0} = \alpha + i\beta \iff \boxed{f'(z_0) = \alpha + i\beta}.$$

which is equivalent to

$$f(z) - f(z_0) = (\alpha + i\beta)(z - z_0) + \underbrace{o(1)}_{\text{tends to zero as } z \to z_0}|z - z_0|$$

which can be rewritten

$$\begin{pmatrix} u \\ iv \end{pmatrix} - \begin{pmatrix} u_0 \\ iv_0 \end{pmatrix} = \underbrace{\Big[(\alpha + i\beta)(\Delta x + i\Delta y) \Big]}_{\begin{pmatrix} \alpha\Delta x - \beta\Delta y \\ i(\beta\Delta x + \alpha\Delta y) \end{pmatrix}} + o(1)\underbrace{|\Delta z|}_{\sqrt{\Delta x^2 + \Delta y^2} = |\underset{\sim}{x} - \underset{\sim}{x}_0|}$$

Conclude: To show $f'(z_0)$ exists we need only find (α, β) such that

$$\begin{pmatrix} u \\ iv \end{pmatrix} - \begin{pmatrix} u_0 \\ iv_0 \end{pmatrix} - \begin{bmatrix} \alpha & -\beta \\ i\beta & i\alpha \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = o(1)\,|\underset{\sim}{x} - \underset{\sim}{x}_0|.$$

This alone implies $f'(z_0) = \alpha + i\beta$.

But assuming u, v differentiable at $\underset{\sim}{x}_0$ implies

$$\begin{pmatrix} u \\ iv \end{pmatrix} - \begin{pmatrix} u_0 \\ iv_0 \end{pmatrix} = \begin{bmatrix} a & b \\ ic & id \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} + o(1)\,|\underset{\sim}{x} - \underset{\sim}{x}_0|$$

put i's in to make comparison

That is – we need an $(\alpha,\beta)\in\mathbb{R}^2$ such that

(1) $$\begin{pmatrix} u \\ iv \end{pmatrix} - \begin{pmatrix} u_0 \\ iv_0 \end{pmatrix} = \begin{bmatrix} \alpha & -\beta \\ i\beta & i\alpha \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = o(1)\,|\underset{\sim}{x}-\underset{\sim}{x}_0|.$$

and assuming differentiability of u, v at $\underset{\sim}{x}_0$ we have

(2) $$\begin{pmatrix} u \\ iv \end{pmatrix} - \begin{pmatrix} u_0 \\ iv_0 \end{pmatrix} = \begin{bmatrix} a & b \\ ic & id \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} + o(1)\,|\underset{\sim}{x}-\underset{\sim}{x}_0|$$

with $a = u_x(\underset{\sim}{x}_0)$, $b = u_y(\underset{\sim}{x}_0)$, $c = v_x(\underset{\sim}{x}_0)$, $d = v_y(\underset{\sim}{x}_0)$.

Finally, if we assume (CR) holds at $\underset{\sim}{x}_0$,
then $a=d$, $b=-c$, and (2) becomes

(2)′ $$\begin{pmatrix} u \\ iv \end{pmatrix} - \begin{pmatrix} u_0 \\ iv_0 \end{pmatrix} = \begin{bmatrix} a & b \\ -ib & ia \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} + o(1)\,|\underset{\sim}{x}-\underset{\sim}{x}_0|$$

So (CR) implies (1) holds with $\alpha = a$, $\beta = -b$, in which case $f'(z_0)$ exists and $f'(z_0) = \alpha + i\beta$.

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