III Complex Exponential Function

- Recall that the real exponential $y = e^x$ and the trigonometric functions $y = \sin x$, $y = \cos x$ are given by globally convergent power series which look mysteriously similar –

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sin x = \sum_{n=1}^{\infty} \underbrace{(-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}}_{\substack{\text{odd terms from } e^x \\ \text{alternating in sign}}}, \quad \cos x = \sum_{n=0}^{\infty} \underbrace{(-1)^n \frac{x^{2n}}{(2n)!}}_{\substack{\text{even terms from } e^x \\ \text{alternating in sign}}}$$

- Recall that you can differentiate the power series term by term (TxT) because the convergence is <u>uniform</u>:

$$\frac{d}{dx} e^x = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{x^n}{n!} = \sum_{n=0}^{\infty} n \frac{x^{n-1}}{n!} \underset{k=n-1}{=} \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

$$\frac{d}{dx} \sin x = \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} (-1)^n (2n-1) \frac{x^{2n-2}}{(2n-1)!} = \sum_{k=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= \cos x$$

$$\frac{d}{dx} \cos x = \frac{d}{dx} \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \overset{(HW)}{=} \ldots = -\sin x$$

• Since $\frac{d}{dx} x^n = n x^{n-1}$ (real) $\sim$ $\frac{d}{dz} z^n = n z^{n-1}$ (complex)

it makes sense that the complex exponential should be defined by

$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ so that

$$\frac{d}{dz} e^z = \frac{d}{dz} \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{d}{dz} \frac{z^n}{n!} = \sum_{n=0}^{\infty} n \frac{z^{n-1}}{n!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

Turns out: all of this makes sense so long as the power series converges uniformly (which it does - later)

• Similarly, it makes sense to define

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n-1}}{(2n-1)!}, \quad \cos z = \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \qquad (*)$$

(HW) Show that if you can differentiate (*) T×T then the following hold in the complex sense:

"if e^z, $\cos z$, $\sin z$ were defined by power series"

$$\frac{d}{dz} \sin z = \sum_{n=0}^{\infty} \frac{d}{dz} (-1)^n \frac{z^{2n-1}}{(2n-1)!} = \cos z$$

$$\frac{d}{dz} \cos z = \sum_{n=1}^{\infty} \frac{d}{dz} (-1)^n \frac{z^{2n}}{(2n)!} = -\sin z$$

- Consider then what we would get if we put $z = iy$ into the power series for e^z:

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{n=0}^{\infty} \frac{(i)^n y^n}{n!} =$$ (separate into terms w n even & odd)

$$= \underbrace{\sum_{n=0}^{\infty} (i)^{2n} \frac{y^{2n}}{(2n)!}}_{\text{even powers}} + \underbrace{\sum_{n=1}^{\infty} (i)^{2n-1} \frac{y^{2n-1}}{(2n-1)!}}_{\text{odd powers}}.$$

(HW) convince yourself this is correct

Now: $(i)^{2n} = (i^2)^n = (-1)^n$;

$$(i)^{2n-1} = (i)^{2n}(i)^{-1} = (-1)^n(-i) = (-1)^{n+1} i$$

Conclude:

$$e^{iy} = \cos y + i \sin y$$

- Although all of this is motivated by the conjecture that e^z can be defined by a power series, the easiest way to develop the theory is to start with the definition

Defn: $e^z = e^{x+iy} \equiv e^x e^{iy} = e^x(\cos y + i \sin y)$

(This defn does not require a theory of power series)

- **Theorem:** $e^z = e^x(\cos y + i\sin y)$ defines a complex valued function which is entire in the sense that its derivative exists $\forall z \in \mathbb{C}$

Proof: It suffices to check that the Cauchy-Riemann equations hold (CR): $u_x = v_y$, $u_y = -v_x$

For this write:

$$e^z = \underbrace{e^x \cos y}_{u(x,y)} + i \underbrace{e^x \sin y}_{v(x,y)}$$

$u(x,y) = e^x \cos y$ | $v(x,y) = e^x \sin y$

$u_x = e^x \cos y$, $u_y = -e^x \sin y$ | $v_x = e^x \sin y$, $v_y = e^x \cos y$

So: $\boxed{u_x = e^x \cos y = v_y \quad u_y = -e^x \sin y = -v_x}$

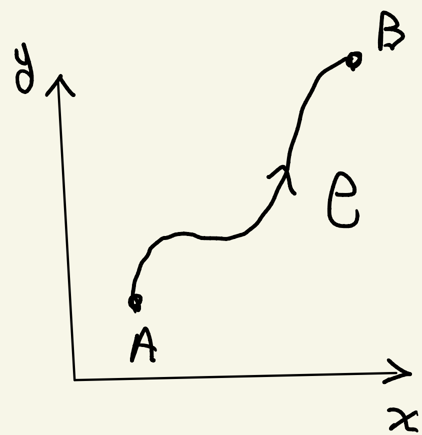
$\checkmark$ (CR)

Moreover: Since $f(z) = e^z$ satisfies (CR) $\forall z \in \mathbb{C}$, it follows that $f'(z)$ exist indept of $\Delta z \to 0$ for every z, and $f'(z) \underset{\uparrow}{=} u_x + i v_x = e^z$

$\Delta z = \Delta x$

• **Conclude**: By defining $e^z = e^x e^{iy}$ with $e^{iy} = \cos y + i \sin y$, we can verify the (CR) equations and $\frac{d}{dz} e^z = e^z$ directly, w/o having to appeal to power series at all!

Our theory tells us that e^z is an entire function whose derivative and anti-deriv agree with e^z, so



$$\int_C e^z dz = e^z \Big]_A^B = e^B - e^A$$

$$\oint_C e^z dz = 0 \quad (A = B)$$

Theorem: With this definition, the complex exponential satisfies the fundamental properties of the real exponential (HW) Proof

(i) $e^{z+w} = e^z e^w$ (iii) $|e^x e^{iy}| = e^x$

(ii) $e^z \neq 0$ (iv) $\overline{e^z} = e^{\bar{z}}$

- Note the proof of (i) requires establishing

$$e^{i(y_1+y_2)} = e^{iy_1}e^{iy_2}$$

which says (remarkably)

$$\cos(y_1+y_2)+i\sin(y_1+y_2) = (\cos y_1 + i\sin y_1)\cdot(\cos y_2 + i\sin y_2)$$

a wonderful identity which follows from the sum/angle identities for sin/cos.

An immediate Corollary is:

Theorem: (De Moire's Formula)

$$(e^{iy})^n = e^{iny} \quad \text{or} \quad \boxed{(\cos y + i\sin y)^n = \cos ny + i\sin ny}$$

- Essentially, De Moire's Formula uses complex numbers to organize the sum angle formulas of trigonometry into simple complex multiplication: $(e^{iy})^n = e^{iny}$!

This turns out to make Fourier Series much simpler with complex numbers, and so complex variables is the language of signal processing.

- **Note:** The problem with defining the exponential by its power series, aside from requiring the theory of power series to do it, is verifying

$$e^{z+w} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) = e^z e^w$$

By defining $e^z = e^x e^{iy}$ and $e^{iy} = \cos y + i \sin y$, we can verify $e^{z+w} = e^z e^w$ by trig identities, and then we can use the theory of power series to prove $e^x(\cos y + i \sin y) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

In particular: Uniform convergence of series $\overset{\text{implies}}{\Longrightarrow}$ TxT differentiation $\overset{\text{implies}}{\Longrightarrow}$ $\frac{d}{dz} \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Thus both $e^z = e^x(\cos\theta + i \sin\theta)$ & $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ solve the ordinary differential equation (ODE)

$$\boxed{\begin{aligned} f'(z) &= f(z), \\ f(0) &= 1 \end{aligned}}$$

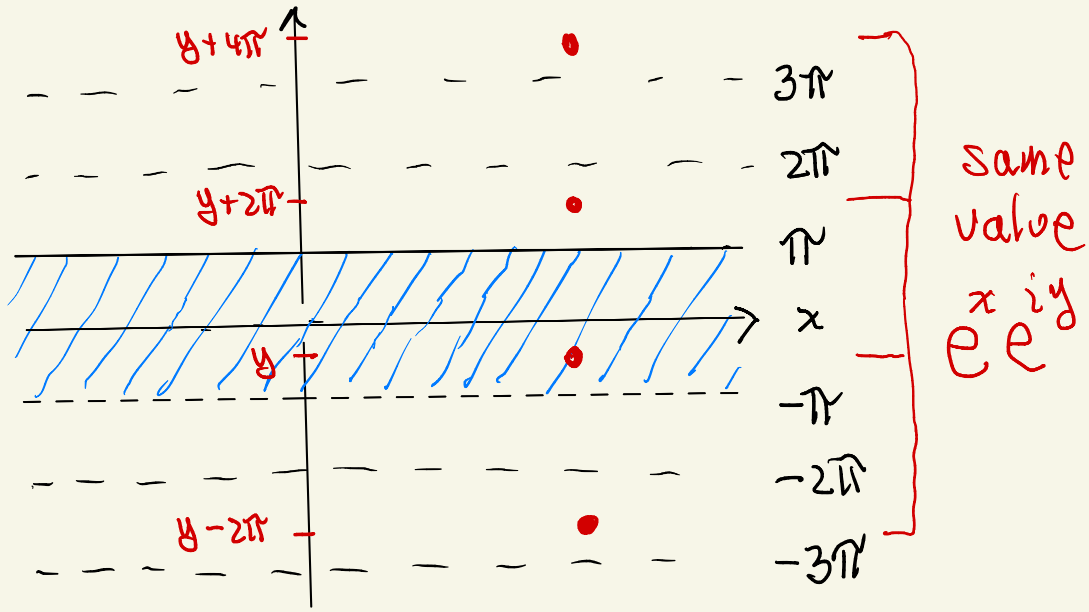
and the theory of ODE's $\overset{\text{implies}}{\Longrightarrow}$ this initial value problem has precisely one soln $\Rightarrow$ $\boxed{e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}}$

- **Corollary:** $e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

implies $\Rightarrow$ e^z is **2π-periodic** in y,

$$\boxed{e^{x+iy} = e^{x+i(y+2\pi n)}}$$

Picture:



"All values of e^z are taken on in the strip $-\infty < x < \infty$, $-\pi < y < \pi$, and no value of e^z is taken on twice in this strip, i.e., $e^{z_1} = e^{z_2}$, $z_1, z_2 \in S = \{z : -\infty < x < \infty, -\pi < y \leq \pi\}$ iff $z_1 = z_2$." (HW: prove this)

- **Ex:** $e^{i\pi} = \cos\pi + i\sin\pi = \cos\pi = -1$

$$\boxed{e^{i\pi} + 1 = 0}$$

relates all of the fundamental numbers of classical math!

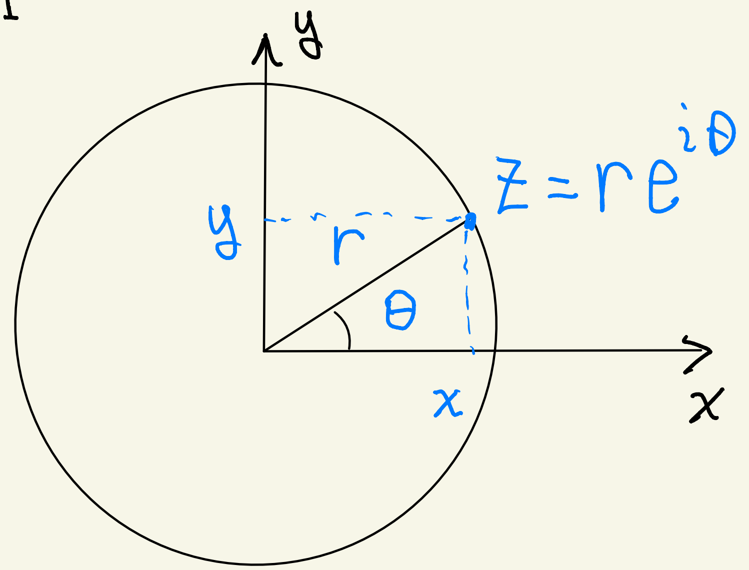
☒ The exponential gives us a way to interpret complex multiplication geometrically:

- For $z = x + iy$, we can write $r = |z|$ and

$$z = \underbrace{r}_{\text{length}} e^{i\theta} = r\underbrace{(\cos\theta + i\sin\theta)}_{\substack{\text{unit vector at angle } \theta \\ \text{or a point on unit circle}}}$$

$$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$|re^{i\theta}| = |r||e^{i\theta}| = r$$



Conclude: $|z| = r$

$$r = \sqrt{x^2 + y^2}$$

Theorem: Every complex number $z = x + iy$ can be uniquely expressed as $z = re^{i\theta}$ where $r = |z|$ & θ is its angle with the x-axis, say $0 < \theta < \pi$ (or any other 2π interval of θ)

$z = re^{i\theta}$ says $x = r\cos\theta$
$y = r\sin\theta$

So:

$$z = \underbrace{r}_{r = \text{modulus}} e^{i\theta} = |z| \underbrace{(\cos\theta + i\sin\theta)}_{\substack{(\cos\theta,\ \sin\theta) \\ \text{unit direction}}}$$

θ = angle

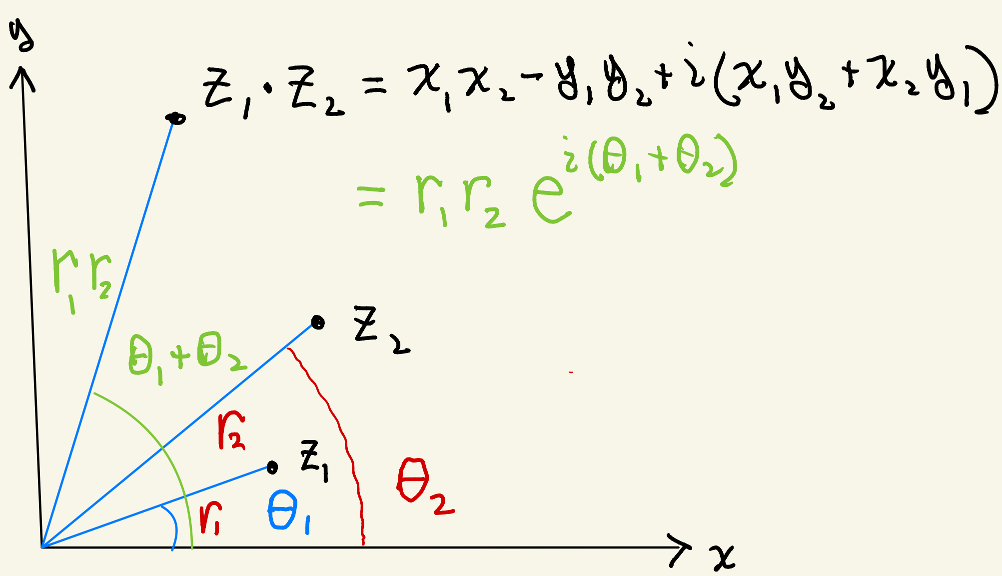
- This provides a **geometric interpretation** for complex multiplication: $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

$$z_1 \cdot z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

$$z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = \underbrace{r_1 r_2}_{\substack{\text{multiply} \\ \text{modulus}}} e^{i\underbrace{(\theta_1+\theta_2)}_{\substack{\text{add} \\ \text{angles}}}}$$

Conclude: Complex multiplication **multiplies** the **length** and **adds** the **angles** of two vectors.

Picture:



Q: How can we extend $\cos x$ & $\sin x$ to complex valued functions $\cos z$, $\sin z$ w/o appealing to power series?

Ans: Use $e^{ix} = \cos x + i \sin x$, $x \in \mathbb{R}$

Lemma: For real $x \in \mathbb{R}$ we have

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Proof: (HW) i.e,

$$\frac{e^{ix} - e^{-ix}}{2i} = \frac{\cancel{\cos x} + i \sin x - \cancel{\cos(-x)} - i \sin(-x)}{2i} = \frac{2i \sin x}{2i} = \sin x$$

cos-even
sin-odd

$$\frac{e^{ix} + e^{-ix}}{2} = \text{(HW)} = \cos x$$

- From this it makes sense to define

Defn: $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

Claim: This correctly extends $\sin x$ & $\cos x$ to complex functions which satisfy CR —

Defn: An entire function is one which is has a complex derivative $\forall z \in \mathbb{C}$.

Theorem: $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ & $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ satisfy the CR equations at every $z \in \mathbb{C}$ and hence are entire.

Proof: Consider first $f(z) = \cos z$

$$f(z) = \cos z = \frac{e^{iz}+e^{-iz}}{2} = \frac{e^{i(x+iy)}+e^{-i(x+iy)}}{2}$$

$$= \frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos(-x) + i\sin(-x))}{2}$$

$$= \frac{1}{2}\left(e^{-y}\cos x + e^{y}\cos x\right) + i\left(e^{-y}\sin x - e^{y}\sin x\right)$$

$$= \underbrace{\frac{e^{-y}+e^{y}}{2}\cos x}_{u(x,y)} + i\,\underbrace{\frac{e^{-y}-e^{y}}{2}\sin x}_{v(x,y)}$$

Thus: $u_x = -\frac{e^{-y}+e^{y}}{2}\sin x$, $v_x = \frac{e^{-y}-e^{y}}{2}\cos x$

$u_y = \frac{-e^{-y}+e^{y}}{2}\cos x$, $v_y = \frac{-e^{-y}-e^{y}}{2}\sin x$

and hence $u_x = v_y$, $u_y = -v_x$ ✓ CR

Similarly, (HW)

$$\sin z = \underbrace{\frac{e^{y}+e^{-y}}{2}\sin x}_{u(x,y)} + i\underbrace{\frac{e^{y}-e^{-y}}{2}\cos x}_{v(x,y)}$$

and $u_x = v_y$, $u_y = -v_x$ (Check HW)

Conclude: $\cos z$ and $\sin z$ are entire.

• Note: $$\cos z = \underbrace{\frac{e^{-y}+e^{y}}{2}}\cos x - i\underbrace{\frac{e^{y}-e^{-y}}{2}}\sin x$$

$$= \cosh y \cos x - i \sinh y \sin x$$

$$\sin z = \frac{e^{y}+e^{-y}}{2}\sin x + i\,\frac{e^{y}-e^{-y}}{2}\cos x$$

$$= \cosh y \sin x + i \sinh y \cos x$$

So (CR) follows by

$$\frac{d}{dx}\underbrace{\cosh x}_{\text{hyperbolic cosine}} = \sinh x, \quad \frac{d}{dx}\underbrace{\sinh x}_{\text{hyperbolic sine}} = \cosh x$$

Since $\cos z$ and $\sin z$ satisfy CR, we know $\frac{d}{dz}\cos z$ and $\frac{d}{dz}\sin z$ exist. We calculate them as follows:

Theorem: $\frac{d}{dz}\cos z = -\sin z$, $\frac{d}{dz}\sin z = \cos z$

(1) $\frac{d}{dz}\cos z \underset{\Delta z = \Delta x}{=} u_x + i v_x$

$$= \frac{\partial}{\partial x}(\cosh y \sin x) + i\frac{\partial}{\partial x}(-\sinh y \sin x)$$

$$= -\cosh y \sin x - i \sinh y \cos x$$

$$= -\sin z$$

(2) $$\frac{d}{dz}\frac{e^{iz}+e^{-iz}}{2} = \frac{ie^{iz}-ie^{-iz}}{2} = i\,\frac{e^{iz}-e^{-iz}}{2i}\,i = -\frac{e^{iz}-e^{-iz}}{2i}$$

$$= -\sin z$$

(HW) Check $\frac{d}{dz}\sin z = \cos z$ both ways... ✓

✓

• Note: We used Chain Rule: Assume f', g' exist.

then $\frac{d}{dz}f(g(z)) = f'(w)\,g'(z)$, $w = g(z)$,

So: $$\frac{d}{dz}e^{g(z)} = \frac{d}{dw}e^{w}\,\frac{dw}{dz} = e^{g(z)}g'(z)$$

The Complex Logarithm –

We found that $F(z) = \ln|z| + i\,\theta(z)$ is the anti-derivative of $f(z) = \frac{1}{z}$, but it is only defined when the neg x-axis (or any $\frac{1}{2}$-line) is removed to make $D' = \mathbb{C} \setminus \{\text{neg } x\text{-axis}\}$ a simply connected domain. Here (with $\theta \in (-\pi, \pi]$)

$$\theta = \operatorname{Arctan}\left(\frac{y}{x}\right) = \operatorname{Arccot}\left(\frac{x}{y}\right)$$ (in case $x = 0$ or $y = 0$)

Our Theory tells us $F(z)$ is analytic in D'.

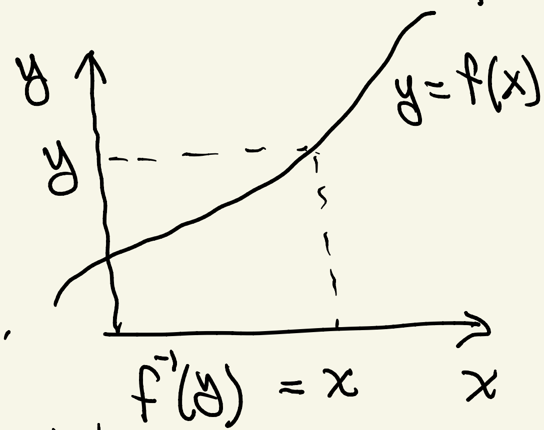
We now show that $F(z) = \ln|z| + \theta(z)$ is the inverse of $w = e^z$, and we name it:

Defn: $\operatorname{Log} z = \ln|z| + i\,\theta(z) \qquad z \in D'$

Defn: The axis removed to make D' simply connected is called a "branch cut"

We now show that on its domain D', $w = \operatorname{Log} z$ is the inverse of $w = e^z$.

• **Recall**: a function $y = f(x)$, $f: \mathbb{R} \to \mathbb{R}$ (real case) has an inverse if you can solve for $x = f^{-1}(y)$, which you can do if $f'(x) \neq 0$.

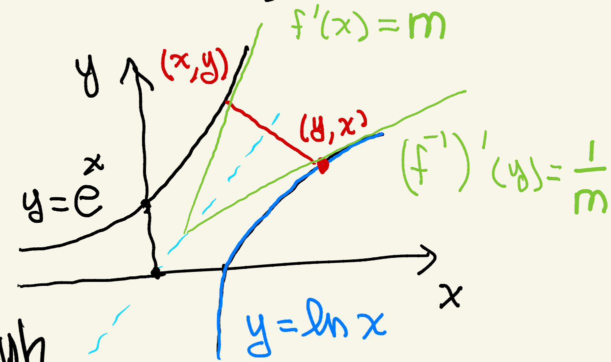


Then $y = f(x)$ iff $x = f^{-1}(y)$

and $f(f^{-1}(y)) = y$, $f^{-1}(f(x)) = x$.

You have to be more careful if either the Domain of f, or its Range, is not all of $\mathbb{R}$.

Ex: $y = e^x$ is defined for all $x \in \mathbb{R}$, but its range is only $y > 0$.



$y = e^x$ iff $x = \ln y$

(x, y) on graph iff (y, x) on graph

$e^x : \mathbb{R} \to \mathbb{R}^+$ $\quad \ln x : \underbrace{\mathbb{R}^+}_{\text{restricted Domain}} \to \mathbb{R}$

With these restrictions –

$$\ln e^x = x, \quad y = e^x$$

Chain Rule: $\dfrac{d\ln y}{dy} \underbrace{\dfrac{dy}{dx}}_{e^x = y} = 1 \Rightarrow \boxed{\dfrac{d\ln y}{dy} = \dfrac{1}{y}}$

- For the complex exponential its more complicated to find the inverse because

$$\exp(z) = e^{z} \qquad \exp : \mathbb{C} \longrightarrow \mathbb{C}$$

the image covers $\mathbb{C}$ multiple times. thus we proceed more generally –

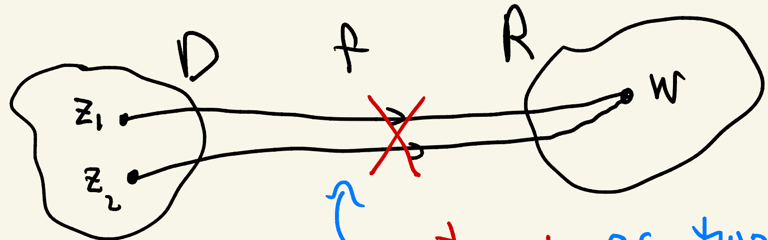
Assume $f : D \longrightarrow R$ $\quad D \equiv$ Domain, $R \equiv$ Range

$$D, R \subseteq \mathbb{C}$$

<u>Defn</u>: We say f is 1-1 "one to one" if no two inputs hit the same output. I.e.,

$$f(z_1) = f(z_2) \underset{\text{implies}}{\Longrightarrow} z_1 = z_2$$

<u>Picture</u>:



not 1-1 as two inputs hit same output w

1-1 says every w in the image of D

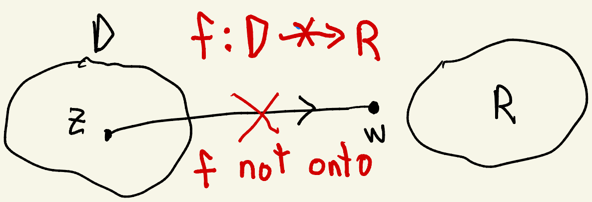
$$w \in f(D) = \{ w : \exists\, z \text{ st } f(z) = w \}$$

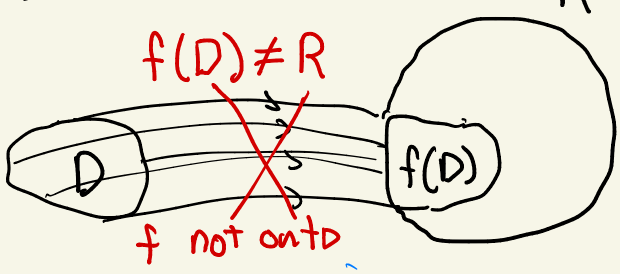
"the image of D"

has a <u>unique</u> <u>pre-image</u> to define the inverse.

Defn: $f: D \to R$ is onto R if every $z \in R$ is in the image of f, i.e.

f onto if for every $w \in R$ there exists $z \in D$ such that $f(z) = w$.

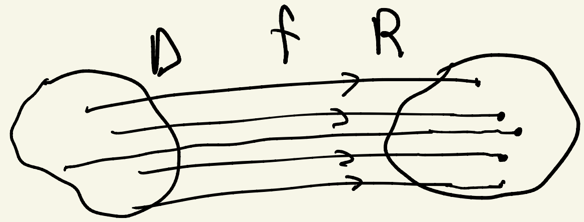




"f onto means every $w \in R$ is hit by some $z \in D$ under f
$\equiv R = f(D)$ as sets"

Theorem: $f: D \to R$ has a unique inverse $f^{-1}: R \to D$ iff f is 1-1 and onto.

Proof: (HW) You have to show that for every w in R there is exactly one $z \in D$ such that $w = f(z)$, so $z = f^{-1}(w)$ defines the inverse function.



"a 1-1 onto mapping goes both ways"

Theorem: If $w = f(z)$, $f : D \to R$, is 1-1 and onto, and f is differentiable at z, then

$$(f^{-1})'(w) = \frac{1}{f'(z)}$$

Proof: f 1-1, onto $\underset{\text{implies}}{\Rightarrow}$ a unique inverse f^{-1} exists such that $f(f^{-1}(w)) = w$, $f^{-1}(f(z)) = z$.

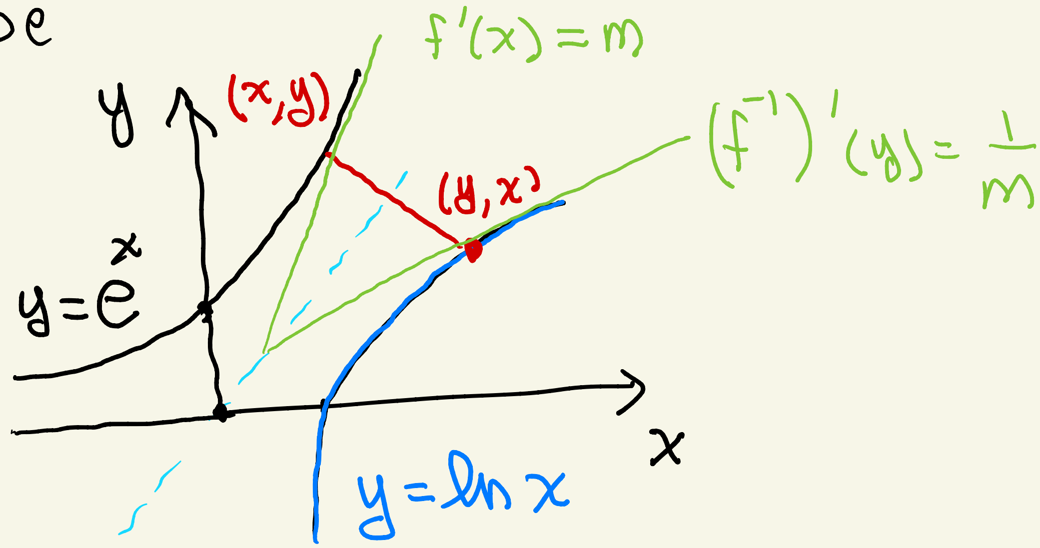
Differentiating both sides of $f^{-1}(f(z)) = z$ gives

$$\frac{d}{dz} f^{-1}(\underbrace{f(z)}_{w}) = \underbrace{(f^{-1})'(w) \frac{dw}{dz}}_{\text{chain rule}} = \frac{d}{dz} z = 1$$

So $\quad (f^{-1})'(w) = \dfrac{1}{(dw/dz)} = \dfrac{1}{f'(z)}$ ✓

at the w & z st $w = f(z)$

Again - can be pictured for $f : \mathbb{R} \to \mathbb{R}$



- Consider now $w = f(z) = e^{z}$

$$u + iv = w = e^{x+iy}$$

To obtain an inverse, we must restrict z to a Domain where f is 1-1, and find the range $R = f(D)$ so $f: D \to R$ 1-1 & onto.

First, $w = e^{z} = \underbrace{e^{x}}_{\text{length}} \underbrace{e^{iy}}_{\text{unit vector}} = r(\cos y + i \sin y) \neq 0$

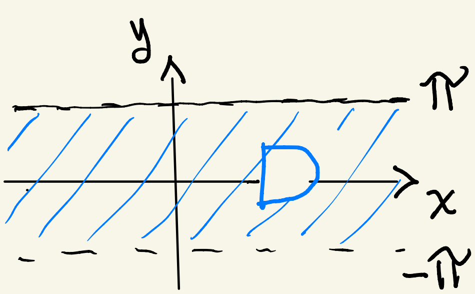
Here: $r = e^{x} > 0$ can be any positive real

$\overrightarrow{(\cos y, \sin y)}$ can be any direction.

Thus the range of $w = e^{z}$ is $R = \mathbb{C} \setminus \{w = 0\}$

Since e^{z} is 2π-periodic in y, restrict to

$$D = \{ z \in \mathbb{C} : x \in \mathbb{R},\ -\pi < y \leq \pi \}$$



$f(z) = e^{x} e^{iy}$

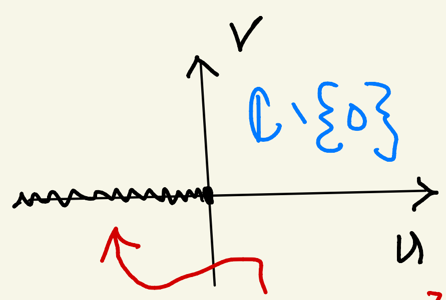
$f(D) = \mathbb{C} \setminus \{w = 0\}$

1-1

onto

$f^{-1}(w) = x + iy$

$f^{-1}(\mathbb{C} \setminus \{0\}) = D$



values of e^{z} don't match at $y = \pi, -\pi$

- **Summary** – To construct the inverse of $f: D \to R$ 1-1 onto ; $D, R \subseteq \mathbb{C}$

Start: $w = f(z)$

Switch z, w (to make z the input of f^{-1}): $z = f(w)$

Solve for w: $w = f^{-1}(z)$, $f^{-1}: R \to D$

- So start with $u + iv = e^x e^{iy} \iff w = e^z$

Switch z, w: $x + iy = e^u e^{iv} \iff z = e^w$

Solve for w: $|z| = e^u \iff u = \ln|z|$

$$z = \underbrace{r}_{\text{length}} \underbrace{e^{iv}}_{\text{unit direction}} \Rightarrow v = \theta(z) = \text{the angle } z \text{ makes with } x\text{-axis}$$

$$\theta(z) = \mathrm{Arctan}\left(\frac{y}{x}\right) = \mathrm{Arccot}\left(\frac{x}{y}\right)$$

Conclude: $\boxed{u + iv = \ln|z| + i\theta}$

OR: $\boxed{z = re^{i\theta} \iff w = \ln r + i\theta = \mathrm{Log}\, z}$

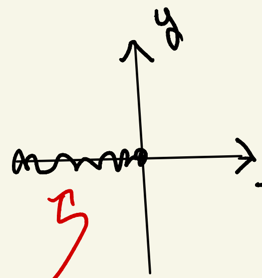
Notation: We let $\ln(r)$ denote the Real logarithm, use $\log z$ to denote complex logarithm, and upper case $\mathrm{Log}\, z$ usually means we've chosen a standard branch –

Define $\boxed{\text{Log } z = \ln r + i\theta}$

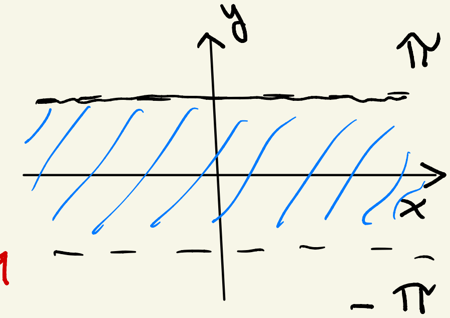
Log: $D \longrightarrow R$

$\mathbb{C}\setminus\{0\}$ $\quad$ $\{w \in \mathbb{C} : -\pi < v \le \pi\}$

Log : $z \longmapsto w$



$w = \text{Log } z$



undefined @ $z=0$
discontinuous along neg real axis

"a simply connected domain D' where $\frac{d \text{Log} z}{dz}$ exists"

strip of height 2π in v

- We already checked CR equations hold with $\theta = \text{Arctan}\, y/x = \text{Arccot}\, \frac{x}{y}$, and we have $\frac{d}{dz}\text{Log } z = \frac{1}{z}$ off the branch cut. We can also get this by chain rule from $\log(e^w) = w$

$$\frac{d}{dw}\text{Log}(e^w) = \frac{d}{dz}\text{Log } z \underbrace{\frac{de^w}{dw}}_{e^w = z} = \frac{d}{dw} w = 1$$

$$\Rightarrow \boxed{\frac{d}{dz}\text{Log } z = \frac{1}{z}}$$

The Inverse Function Theorem –

- we just defined the complex exponential

$$e^{z} = e^{x+iy} \equiv e^{x} e^{iy} = e^{x}(\cos y + i \sin y)$$

and showed that on the Domain

$$D = \{ z : -\pi < \theta \leq \pi \}$$

$f(z) = e^{z}$ is 1-1 and onto the range $R = \mathbb{C} \setminus \{0\}$

(not to be confused with $\mathbb{R} \equiv$ real #s)

Thus its inverse exists for $z \neq 0$

$$f^{-1} : R \longrightarrow D, \quad f^{-1}(z) = \ln|z| + i\theta(z)$$

and the inverse is analytic, with $f^{-1\prime}(z) = \frac{1}{z}$,

except f^{-1} is discontinuous on neg real axis where θ jumps from $-\pi$ to π, and so $(f^{-1})'(w)$ only exists in $D' = \mathbb{C} \setminus \underbrace{\{\text{neg real axis}\}}_{\text{"branch cut"}}$

We defined: $\operatorname{Log} z = f^{-1}(z) = \ln|z| + i\theta(z)$

$$\operatorname{Log} : D' \longrightarrow D,$$

so Log is continuously differentiable in D'.

- Q: Is the inverse of an analytic fn always analytic?

- Lets deconstruct the argument that gives the formula $f^{-1\prime}(w) = \frac{1}{f'(z)}$, $w = f(z)$, $f^{-1}(w) = z$

- First, assume $f : D \to R$ f is analytic (f' exists indept of $\Delta z \to 0 \Leftrightarrow$ CR), 1-1 and onto, so $f^{-1} : R \to D$ exists, $w = f(z)$, $z \in D$, $w \in R$.

- By Defn, $\underbrace{f^{-1}(f(z))} = \underbrace{z}$

  same function of z

- Since both sides name the same function, the derivatives must be equal on both sides

- Differentiate both sides and set them equal

$$\underbrace{\frac{d}{dz} f^{-1}(f(z))} = \frac{d}{dz} z = 1$$

- Chain Rule: $\dfrac{df^{-1}}{dw}\dfrac{dw}{dz} = f^{-1\prime}(w) f'(z) = 1$

so $\quad (f^{-1})'(w) = \dfrac{1}{f'(z)} \leftarrow$ needs $f'(z) \neq 0$

- Problem: We need to know $(f^{-1})'(w)$ exists in order to apply the Chain Rule ?

• So Question: If a complex valued function $f: D \to R$ is analytic in D, and 1-1 onto $R = f(D)$,[†] (so $f^{-1}: R \to D$ exists), is f^{-1} always analytic?

[†] $f(D) = \{w : \exists\, z \in D \text{ st } w = f(z)\}$ = "image of D under f"

Ans: Yes but you need $f'(z) \neq 0$.
This follows from the complex version of the Inverse Function Theorem

Theorem: (Complex IFT): Assume $w = f(z) = u + iv$ where u and v are continuous with continuous derivatives (C') in some nbhd of $z_0 \in \mathbb{C}$. Assume further that $f'(z_0)$ exists and $f'(z_0) \neq 0$ at z_0. Then there exists $\varepsilon > 0$ such that

$f: B_\varepsilon(z_0) \to R = f(B_\varepsilon(0))$ is 1-1 onto, $(f^{-1})'(w_0)$ exists, and $(f^{-1})'(w_0)$ exists and equals $\dfrac{1}{f'(z_0)}$.

• Note: An analytic fn won't be invertible in a nbhd of a point z_0 where $f'(z_0) = 0$, so this is the best you can do. The IFT tells us that whenever a C' analytic fn is invertible (meaning the inverse exists), the inverse is differentiable.

* "nbhd of" = "neighborhood of" = "an open set containing"

- To Start we review the Real IFT.

Thm (IFT Real Case - Math 127B) Assume $\vec{f}: D_{\text{open}} \to R$, $\vec{f}(x,y) = \overrightarrow{(u(x,y), v(x,y))}$, where $\underset{\sim}{u} = (u,v)$ in C' in a nbhd of $\underset{\sim}{x}_0 = \overrightarrow{(x_0, y_0)}$. Assume further that

$$(*) \quad \operatorname{Det} J(x_0,y_0) = \begin{vmatrix} -\nabla u(x_0,y_0)- \\ -\nabla v(x_0,y_0)- \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = (u_x v_y - u_y v_x)\Big|_{\underset{\sim}{x}_0} \neq 0$$

"evaluated at $\underset{\sim}{x}_0$"

Then $\vec{f}$ has an inverse in a nbhd of $\underset{\sim}{u}_0 = \vec{f}(\underset{\sim}{x}_0)$. I.e. $\exists\, \varepsilon > 0$ st $f^{-1}: D' \to R'$ is 1-1 onto, f^{-1} is C', and

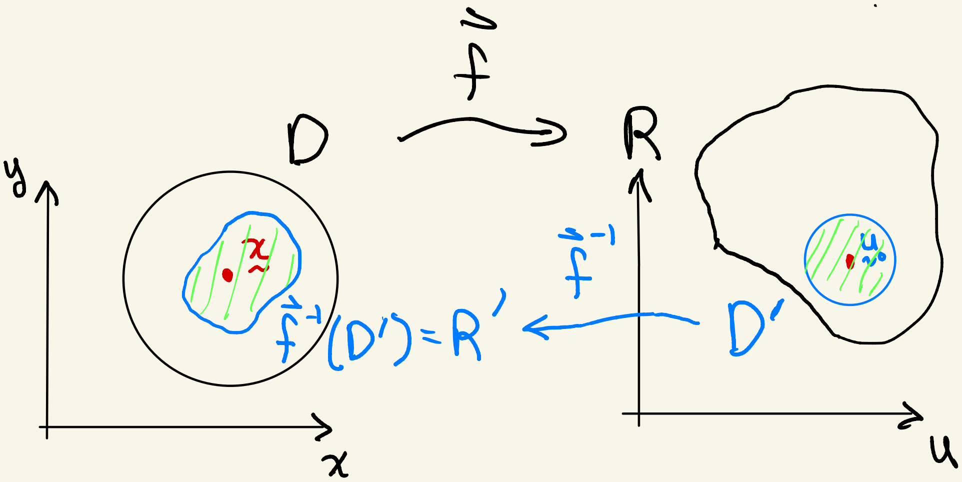
$D' = B_\varepsilon(\underset{\sim}{u}_0)$ $\quad \underset{\sim}{x}_0 \in R'$

$$f^{-1} \circ f(\underset{\sim}{x}) = \underset{\sim}{x} \quad \& \quad f \circ f^{-1}(\underset{\sim}{u}) = \underset{\sim}{u}$$

for $\underset{\sim}{u} \in D'$ and $\underset{\sim}{x} \in R'$.

Proof: (Math 127B)

Figure (*)



- **Conclude**: If $f(z) = u + iv$ is analytic at $z_0 = x_0 + iy_0$, then $\vec{f}(\underset{\sim}{x}_0) = \underset{\sim}{u}_0 = (u_0\ v_0)$ satisfies

$$\text{Det } J(x_0, y_0) = \begin{vmatrix} u_x & u_y \\ -u_y & u_x \end{vmatrix} = u_x^2 + u_y^2 \neq 0$$

$J = D\vec{f}$ ← lines mean take det

Notation: $J(x,y) = Df(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$

iff $f'(z_0) \neq 0$. Thus the condition $f'(z_0)$ exists and is nonzero implies the CR equations, and this in turn guarantees condition (*) of the real IFT. Thus $f'(z_0) \neq 0$ together with $u, v \in C^1$ implies $\vec{f}$ (and hence f) has an inverse in a nbhd D' of $\underset{\sim}{u}_0$

Proof of Complex IFT: Let $f(z) = u(x,y) + iv(x,y)$, and assume $u, v \in C^1$ a nbhd D of z_0, and $f'(z_0) \neq 0$. Now thus CR holds, $u_x = v_y$, $u_y = -v_x$ at z_0. $f'(z_0) = u_x(\underset{\sim}{x}_0) + i u_y(\underset{\sim}{x}_0) \neq 0 \Rightarrow$ either $u_x(\underset{\sim}{x}_0) \neq 0$ or $u_y(\underset{\sim}{x}_0) \neq 0 \Rightarrow \text{Det } J(\underset{\sim}{x}_0) = \begin{vmatrix} u_x & u_y \\ -u_y & u_x \end{vmatrix} = u_x^2 + u_y^2 \neq 0$, and by (*) we know $\vec{f}^{-1} : D' \to R'$ exists (see Fig*). Since $\vec{f}^{-1}(u,v) = (x(u,v), y(u,v))$ is equivalent to $f^{-1}(u+iv) = x(u,v) + iy(u,v)$ as a mapping $(u,v) \to (x,y)$, it follows that $f^{-1}(u+iv) = x(u,v) + iy(u,v)$ is the complex inverse of $f(z)$ on D'.

It remains to prove that $(f^{-1})'(u_0+iv_0)$ exists.

For this it suffices to verify the CR equations.

Let $\underset{\sim}{u} = \overrightarrow{(u,v)}$, $w = u+iv$, so we have

$$w = f(z) \text{ iff } z = f^{-1}(w)$$
$$\overrightarrow{(u,v)} = \vec{f}(x,y) \text{ iff } \overrightarrow{(x,y)} = \vec{f}^{-1}(u,v)$$
$$\underset{\sim}{u} = \vec{f}(\underset{\sim}{x}) \text{ iff } \underset{\sim}{x} = \vec{f}^{-1}(\underset{\sim}{u})$$

does't require $f^{-1\,\prime}(w)$ exists, just $\operatorname{Det} J \neq 0$

Thus $$\vec{f}^{-1}(\vec{f}(\underset{\sim}{x})) = (\vec{f})^{-1} \circ \vec{f}(\underset{\sim}{x}) = \underset{\sim}{x}$$

composition

Diff both sides and apply Chain Rule for $\underline{\text{real}}$ $f: \mathbb{R}^2 - \mathbb{R}^2$:

$$\frac{\partial(\vec{f}^{-1} \circ \vec{f})}{\partial \underset{\sim}{x}} = \frac{\partial \vec{f}^{-1}}{\partial \underset{\sim}{u}} \cdot \frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{x}} = I$$

2x2 2x2 2x2

2x2 identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2x2 matrix of partials

$$\boxed{D\vec{f}^{-1} \cdot D\vec{f} = I} \quad \text{(Chain rule)}$$

Here, $D\vec{f} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = J$, $D\vec{f}^{-1} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$, and

(Chain Rule) says

$$\boxed{D\vec{f}^{-1} = J^{-1}}.$$

2x2 2x2

But recall we have a formula for the inverse of a 2×2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad |A| = ad-bc$$

Check: $A^{-1}A = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$= \frac{1}{ad-bc}\begin{pmatrix} ad-bc, & 0 \\ 0 & , ad-bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \checkmark$$

Thus $D\vec{f}_0^{-1} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$, and (chain rule) gives

$$D\vec{f}^{-1} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}^{-1} = \frac{1}{u_x v_y - u_y v_x}\begin{pmatrix} v_y & -u_y \\ -v_x & u_x \end{pmatrix}$$

So at $z = z_0$

$f'(z_0)$ exists $\Rightarrow$ $u_x = v_y$, $u_y = -v_x$ at $z = z_0$

$$\boxed{\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}_{z=z_0} = \frac{1}{u_x^2 + u_y^2}\begin{pmatrix} u_x & -u_y \\ u_y & u_x \end{pmatrix}_{z=z_0}}$$

Comparing entries at $z = z_0$, where $f'(z_0)$ exists, gives

$$x_u(u_0,v_0) = y_v(u_0,v_0) \quad \& \quad x_v(u_0,v_0) = y_u(u_0,v_0).$$

This says $f^{-1}(w) = x(u,v) + i v(u,v)$ satisfies CR at $u_0 + i v_0$, and hence $f^{-1\prime}(u_0 + i v_0)$ exists $\checkmark$

Conclude: $f^{-1}(w)$ exists and is smooth in a nbhd of $w_0 = u_0 + i v_0$, and $(f^{-1})'(w)$ exists,

so the complex chain rule applies at $z = z_0$:

$$f^{-1}(f(z)) = z$$

thus

$$\frac{d}{dz} f^{-1}(f(z)) = \frac{df^{-1}}{dw}\frac{dw}{dz} = 1 \quad \text{at } z = z_0$$

$$\frac{dw}{dz} = f'(z_0)$$

and thus $\boxed{\dfrac{df^{-1}}{dw}(w_0) = \dfrac{1}{f'(z_0)}}$

This completes the proof of the Complex (IFT)

Summary: If $f: D \to R$ is 1-1, onto, analytic and "smooth" ($u, v \in C^1$), and $f'(z) \neq 0$, then f^{-1} exists, is analytic, and $f^{-1\prime}(w) = \dfrac{1}{f'(z)}$.