IV Complex Exponents

- We define and derive the properties of
$$f(z) = z^a, \quad a \in \mathbb{C}.$$

- We first consider case $a \in \mathbb{R}$

- To start, recall De Moivres Formula

$n \in \mathbb{N}$: $e^{in\theta} = \cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n = (e^{i\theta})^n$

By this we can define z^a for $a \in \mathbb{N}$:
$$z^n = \underbrace{(re^{i\theta})^n}_{\text{product } n \text{ times}} = r^n (e^{i\theta})^n = r^n e^{in\theta}$$

So this defines $f(z) = z^a$ for $a = n \in \mathbb{N}$

$$\boxed{z^n = r^n e^{in\theta}} \quad n \in \mathbb{N}$$

- Consider next $a = 1/n$, $n \in \mathbb{N}$. Then $z^{1/n}$ define n distinct "n'th roots of z".

Defn: We say $w = z^{1/n}$ if $w^n = z$.

For a given $z \in \mathbb{C}$, we now construct all $w \in \mathbb{C}$ such that $w^n = z$. (This gives n solns of $w = z^{\frac{1}{n}}$)

Every other w_k is a repeat of one of these

- Case ①: $z = 1$. Then $w^n = 1 \Rightarrow$

$$w_k = e^{i\frac{2k\pi}{n}}, \quad k = 0, 1, 2, \ldots, n-1$$

I.e., $$w_k^n = \left(e^{i\frac{2k\pi}{n}}\right)^n = e^{i2k\pi} = e^{i(0+2k\pi)} = e^{i\cdot 0} = 1$$

$$e^{iy} = e^{i(y+2k\pi)}$$

Here: $w_k = e^{i\frac{2k\pi}{n}}$, $k = 0, 1, \ldots, n-1$ are the n points equidistant around the unit circle, starting at $w_0 = 1$.

Picture: Let $\Delta\theta = \frac{2\pi}{n}$, $\theta_k = k\Delta\theta$. Then

$w_0 = 1$,

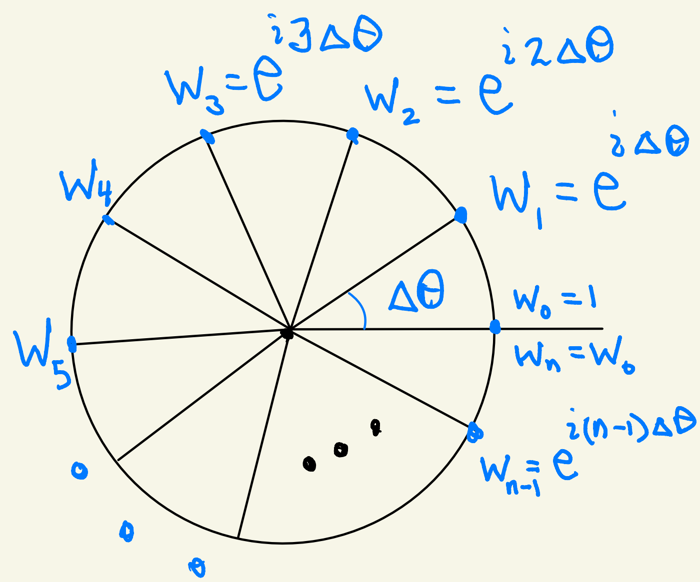
$$w_1 = e^{i\frac{2\pi}{n}} = e^{i\Delta\theta}$$

$$w_2 = e^{i\frac{2\cdot 2\pi}{n}} = e^{i2\Delta\theta}$$

$$\vdots$$

$$w_{n-1} = e^{i\frac{(n-1)\cdot 2\pi}{n}} = e^{i(n-1)\Delta\theta}$$

$$w_n = e^{i\frac{n2\pi}{n}} = e^{i2\pi} = 1 = w_0$$



• Case 2: $z = e^{i\theta}$. Then

$$w^n = e^{i\theta} \Rightarrow w_k = e^{i\left(\frac{\theta+2\pi k}{n}\right)}, \quad k=0,1,\dots,n-1$$

So
$$w_k^n = \left(e^{i\left(\frac{\theta+2\pi k}{n}\right)}\right)^n = e^{i(\theta+2\pi k)} = e^{i\theta} = z$$

$$e^{i(y+2\pi k)} = e^{iy}$$

So $w_k = z^{1/n}$ by Defn.

Here: $w_k = e^{i\frac{2k\pi}{n}}$, $k=0,1,\dots,n-1$ are the n points equidistant around the unit circle, starting at $w_0 = e^{i\theta/n}$

Picture: Let $\Delta\theta = \frac{2\pi}{n}$, $\theta_k = \frac{\theta}{n} + k\Delta\theta$ Then

$$w_0 = e^{i\frac{\theta}{n}}$$

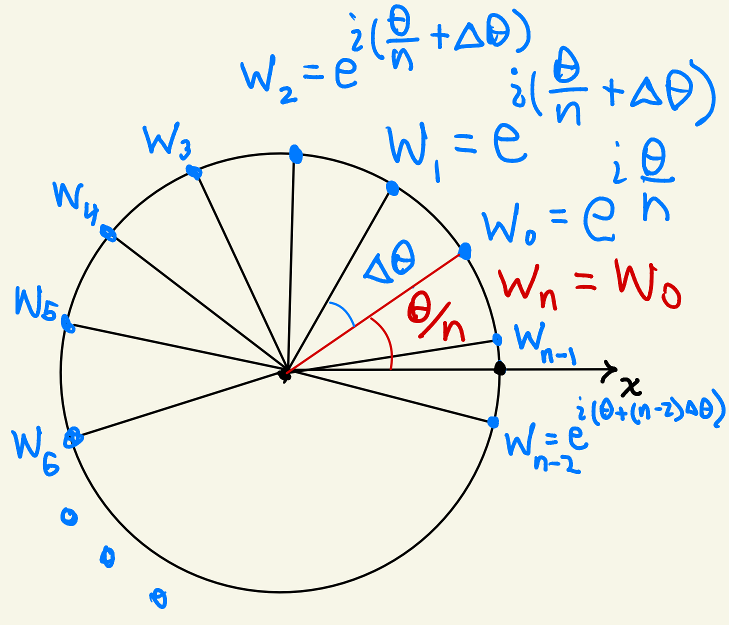
$$w_1 = e^{i\frac{\theta+2\pi}{n}} = e^{i\left(\frac{\theta}{n}+\Delta\theta\right)}$$

$$w_2 = e^{i\frac{\theta+2\cdot 2\pi}{n}} = e^{i\left(\frac{\theta}{n}+2\Delta\theta\right)}$$

$$\vdots$$

$$w_k = e^{i\frac{\theta+k2\pi}{n}} = e^{i\left(\frac{\theta}{n}+k\Delta\theta\right)}$$

$$w_n = e^{i\frac{\theta+n2\pi}{n}} = e^{i\left(\frac{\theta}{n}+2\pi\right)} = e^{i\frac{\theta}{n}} = w_0$$



Defn: $w_n = e^{i\frac{\theta+2k\pi}{n}}$, $n=0,\dots,n-1$ are the n-roots of $e^{i\theta}$

- **Case 3**: $z = re^{i\theta}$. Then

$$w^n = z = re^{i\theta} \Rightarrow w_k = r^{1/n} e^{i\left(\frac{\theta + 2\pi k}{n}\right)}$$

OR:

$$w_k = r^{1/n} \underbrace{e^{i(\theta_0 + k\Delta\theta)}}_{n\text{th roots of } e^{i\theta}}, \quad \theta_0 = \frac{\theta}{n}$$

new length takes it closer to unit circle

(Note: $r^{1/n}$ takes $r>1$ closer to $r=1$, and $r<1$ closer to $r=1$)

Conclude: the n'th roots of $z = re^{i\theta}$ have angles equi-spaced around the unit circle starting at $\theta_0 = \frac{\theta}{n}$, with length $r^{1/n}$, making $z^{1/n}$ closer to unit circle than z.

This defines $w = z^a$ for $a = n, \frac{1}{n}$ $n \in \mathbb{N}$

- **Case 4**: $a = m/n \in \mathbb{Q}_{\text{rationals}}$, $m, n \in \mathbb{Z}_{\text{integers}}$. If $\frac{m}{n} < 0$, define $z^a = \frac{1}{z^{|a|}}$. So wlog, assume $m, n > 0$.

Define: $z^{m/n} = (z^{1/n})^m = (w_0)^m, (w_1)^m, \ldots, (w_{n-1})^m$.

$$w_k = r^{1/n} e^{i\left(\frac{\theta + 2\pi k}{n}\right)} \Rightarrow \boxed{(w_k)^m = r^{m/n} e^{im\left(\frac{\theta + 2\pi k}{n}\right)}}$$

HW: **Conclude**: $w = z^{m/n}$ has n soln's if $\left[\frac{m}{n}\right]$ = lowest terms.

Conclude: $w = z^a$ is defined by explicit construction if $a \in \mathbb{Q}$ real.

Problem ① what if a is irrational?

Eg what is z^{π} or z^{e}? How many roots?

Problem ② What if $a = x + iy \in \mathbb{C}$? How many roots?

For Problem ① we could approx irrational by rational and take limit, like $\pi = 3.14159$, so

$$z^{\frac{314}{100}} \longrightarrow z^{\frac{3141}{1000}} \longrightarrow z^{\frac{31415}{10,000}} \longrightarrow \cdots \longrightarrow z^{\pi}$$

but how do you take limits of all the proliferating roots?

Turns Out: There is a unifying way to treat all complex exponents at once...

- its based on the complex Logarithm
- the branch cuts of the Logarithm take care of the problem of telling us where all the roots of z^a lie $\forall a \in \mathbb{C}$!

General approach to complex exponents — Defining $w = z^a$ for $a \in \mathbb{C}$ using $\operatorname{Log} z$

Defn: $z^a = \left(e^{\operatorname{Log} z}\right)^a = e^{a \operatorname{Log} z}$

$$\boxed{z^a = e^{a \operatorname{Log} z}}$$

We get a different solution for $w = z^a$ for each branch of $\operatorname{Log} z$ we choose —

Eg, if we choose $\operatorname{Log} z = \ln r + i\theta$, $2\pi(k-1) \leq \theta \leq 2\pi k$, $k = 1, 2, 3, \dots, \infty$ we get ALL of the w such that $w = z^a$.

- For $a = m/n$ in lowest terms, the solutions repeat at $k = n$, so we get n distinct soln's.

For irrational a, we get an ∞-# of different solutions, each determind by choice of θ!

(HW) If $a = m/n \in \mathbb{Q}^+$, (the positive rationals), show that the n solutions $w_0, \dots, w_{n-1}$ of $w = z^a$ which we defined above, each correspond to a different brach of $\operatorname{Log} z$ in $w = e^{a \operatorname{Log} z}$.

Solution: Assume $a = m/n \in \mathbb{Q}^+$, & write

$$z^{m/n} \equiv_{\text{Def}} e^{\frac{m}{n}\operatorname{Log} z} = e^{\frac{m}{n}(\ln r + i(\theta + 2k\pi))}$$

a choice of k here is a choice of branch in which $\operatorname{Log} z$ takes values

$$= \underbrace{e^{\frac{m}{n}\ln r}}_{\text{real}} e^{im\frac{\theta+2k\pi}{n}}$$

$$= r^{m/n}\Big(\underbrace{e^{i\frac{\theta+2k\pi}{n}}}\Big)^m$$

These are the n roots $w_0, \dots, w_{n-1}$ of $z = re^{i\theta}$, different for different $k = 0, \dots, n-1$

$$= w_k^m$$

where $w_0^m, \dots, w_{n-1}^m$ are the n solutions of $z^{m/n}$, where $w_0, \dots, w_{n-1}$ are the n'th roots of z.

✓

III Finally, consider the general case $a \in \mathbb{C}$, and define

$$z^a = e^{a \operatorname{Log} z}$$

Each choice of branch for $\operatorname{Log} z$ determines a unique value for z^a, and hence each branch determines a well defined complex valued function

$$w = f(z) = e^{a \operatorname{Log} z}.$$

Since $e^{a \operatorname{Log} z}$ is the composition of analytic functions $f(z) = h \circ g(z)$, $g(z) = a \operatorname{Log} z$, $h(z) = e^z$, where $g(z)$ is analytic off the branch cut for each choice of angle defining $\operatorname{Log} z$

Thus the Chain Rule holds –

$$\frac{d}{dz} z^a = \frac{df}{dz} = \frac{d}{dz} h \circ g = \underbrace{h'(g(z))}_{e^{a \operatorname{Log} z}} \underbrace{g'(z)}_{\frac{a}{z}}$$

$$= \frac{a}{z} e^{a \operatorname{Log} z} = a e^{-\operatorname{Log} z} e^{a \operatorname{Log} z} = a e^{(a-1) \operatorname{Log} z}$$

$$= a z^{a-1} \checkmark$$

(HW) check that all of this makes sense in a single valued sense once branch for $\operatorname{Log} z$ is chosen?

Note on how this works —

- Recall $w = \log z$ is the inverse of $w = e^z$.
  To define $\log z = \ln r + \theta(z)$ we have to choose a branch for $\theta(z)$, $w = \text{Log}\, z$ (usually) taken to mean $-\pi < \theta(z) \le \pi$, differentiable on $-\pi < \theta(z) < \pi$, and given explicity by

$$\theta(z) = \text{Arctan}\, y/x = \text{Arccot}\, x/y\,.$$

- Now $\text{Log}\, z$ and e^z being inverses means

$$e^{\text{Log}\, z} = z = \text{Log}\, e^z, \quad -\pi < \theta \le \pi.$$

But Note: the second one, $\text{Log}\, e^z = z$ requires a branch cut to determine $\text{Log}\, w$, but the first one does not —

$$e^{\log z} = z$$

since the exponential maps all the choices of angle for $\log z$ to the same value, $e^{\log z}$ is indept of branch

That is – $\log z$ is many valued, and we can write

$$\log z = \ln r + i(\underset{\sim}{\theta_0} + 2\pi \underset{\sim}{k})$$

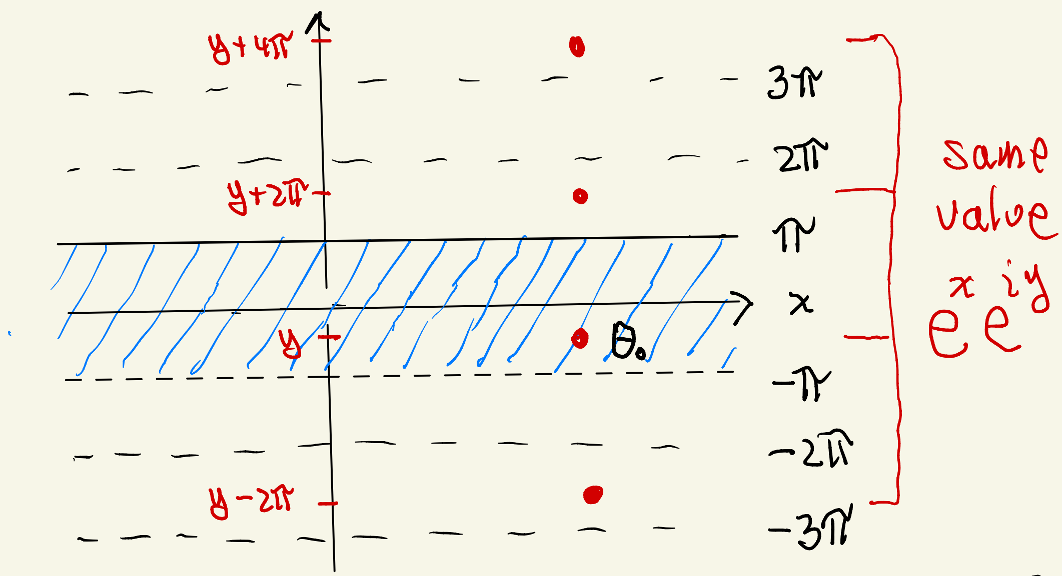
fixed by a branch choice

$k \in \mathbb{Z}$ determines all possible other branch choices

But

$$\boxed{e^{\log z} = e^{\ln r + i(\theta_0 + 2\pi k)} = r e^{i(\theta_0 + 2\pi k)} = z}$$

Picture



Said differently – if we name all the points w_k which e^w maps to the same z, we get

$$w_k = \log z = \underbrace{\ln r}_{\text{indep't of } k} + i\underbrace{(\theta_0 + 2\pi k)}_{\text{all the angles that name } z}$$

and

$$\boxed{e^{w_k} = z}$$ for every $k \in \mathbb{Z}_{\text{integers}}$

Conclude: When we write

$$z^a = e^{a \log z} = e^{a\underbrace{(\ln r + i(\theta_0 + 2\pi k))}_{w_k}}$$

Multiplication of w_k by $a \in \mathbb{C}$ changes all the $w_k \Rightarrow$ either:

(1) $\{a w_k\}_{k=-\infty}^{\infty}$ is a different complex number $\forall k \in \mathbb{Z}$

or

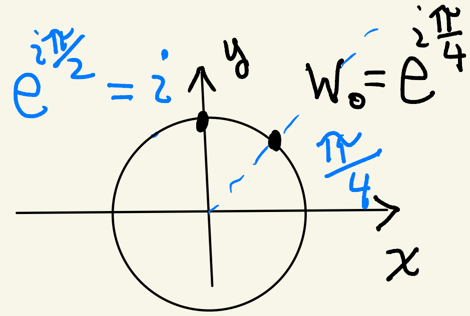
(2) $\{a w_k\}_{k=-\infty}^{\infty}$ is a finite set of numbers because the angles are commensurate with 2π.

Theorem: If $a = \frac{m}{n} \in \mathbb{Q}$, then $\{a w_k\}_{k=-\infty}^{\infty}$ generates a finite # of angles $\theta = \frac{m}{n} 2\pi k$, and if $[\frac{m}{n}] = 1$ lowest terms, then exactly n angles.

If $a \in \mathbb{R}$ irrational, (like $a = \pi, e, \sqrt{2}$), then $a w_k$ generates an infinite number of distinct values $a w_k$, giving k independent branches $z^a = e^{a w_k}$. (Any Choice $\Rightarrow \frac{d}{dz} z^a = a z^{a-1}$).

(HW) Show that our two ways of defining $w = \sqrt{i}, \sqrt[3]{1+i}$ give the same two values –

<u>Soln</u>: 1st $\quad w = (i)^{1/2} = \left(e^{i\frac{\pi}{2}}\right)^{1/2}$



$\Leftrightarrow$ $\quad w^2 = e^{i\pi/2}$

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$\Leftrightarrow$ $\quad w_0 = e^{i\pi/4}, \quad w_1 = e^{i\frac{\pi/2 + 2\pi}{2}} = e^{i\,5\pi/4} = e^{i\left(\frac{\pi}{4}+\pi\right)}$

$w_2 = e^{i\frac{\pi/2 + 4\pi}{2}} = e^{i\frac{9\pi}{4}} = e^{i\left(\frac{\pi}{4}+2\pi\right)} = w_0$

Check: this agrees with $w = e^{a \log z}$, $a = \frac{1}{2}$

$$w = z^{1/2} = e^{\frac{1}{2}\log z} = e^{\frac{1}{2}\log i} = e^{\frac{1}{2}\left(\ln 1 + i\left(\frac{\pi}{2} + 2k\pi\right)\right)}$$

$$= e^{\frac{1}{2} i\left(\frac{\pi}{2} + 2k\pi\right)} = e^{i\left(\frac{\pi}{4} + k\pi\right)} = e^{i\pi/4},\ e^{i\left(\frac{\pi}{4}+\pi\right)}$$

<u>Conclude</u>: $z^{1/2} = e^{\frac{1}{2}\log z}$ defines each branch of the square root in terms of the branch you choose for $\log z$!

In general – when working with complex exponents, obtain a single valued function z^a by simply choosing a branch $\operatorname{Log} z$ and writing $z = e^{a \operatorname{Log} z}$!