Calculating Residues

Recall: f analytic in a nbhd of z_0 $\Rightarrow$

- **Taylor Series**: $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$,

$$c_n = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{(w-z_0)^{n+1}}\, dw = \frac{f^{(n)}(z_0)}{n!}$$

Converges in $B_R(z_0) = \{ z \in \mathbb{C} : 0 \le |z-z_0| < R \}$, where R = distance from $z = z_0$ to nearest singularity in f.

Converges uniformly on every compact subset of $B_R(z_0)$.

- **Laurent Series**: $f(z) = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-z_0)^n} + \sum_{k=0}^{\infty} c_n (z-z_0)^n$

$$c_n = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{(w-z_0)^{n+1}}\, dw$$

(Same formula but applies to $n<0$, but $\nexists\, f^{(n)}(z_0)$)

Converges in the largest annulus

$$B_{r,R}(z_0) = \{ z \in \mathbb{C} : r < |z-z_0| < R \}$$

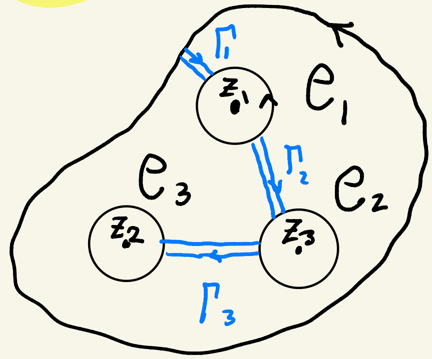
in which f is analytic.

Converges uniformly on every compact subset of $B_{r,R}(z_0)$.

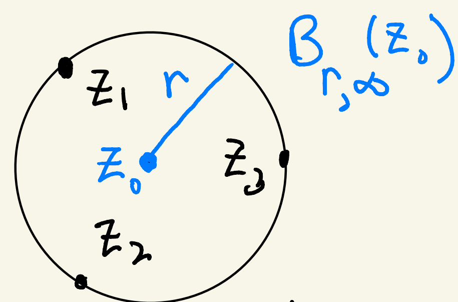
- **Residue Theorem:** If f has point singularities at $z_1, \dots, z_n$, then

$$\oint_{\mathcal{C}} f(z)\,dz = \sum_{k=1}^{n} \oint_{\mathcal{C}_i} f(z)\,dz = 2\pi i \sum_{k=1}^{n} R(f, z_i)$$

$R(f, z_i) = c_{-1}$ in the Larent expansion of f at $z = z_i$.



- **Note:** Our Theory tells us that



f has a Laurent expansion in the annulus $B_{r,\infty}(z_0)$ outside the smallest ball $B_r(z_0)$ which contains all the singularities of f, and $\int_{\mathcal{C}} f(z)\,dz = 2\pi i\, c_{-1}$ for c_{-1} in this expansion. But we have no way to compute c_{-1} in this expansion!

- We now discuss ways to calculate c_{-1} at isolated singularities of meromorphic functions — the case every singularity is a pole, i.e., isolated singularity of finite negative order.

Calculating residues at poles:

- **Simple pole:** z_0 a simple pole of f if

$$f(z) = \frac{c_{-1}}{z-z_0} + \sum_{k=0}^{\infty} c_n (z-z_0)^k, \quad c_{-k}=0, \quad k \geq 2$$

Thm ① f has a simple pole at $z = z_0$ iff

$$\lim_{z \to z_0} f(z)(z-z_0) = L \neq 0,$$

in which case $L = c_{-1}$

All you have to check is $\lim_{z \to z_0} f(z)(z-z_0) = L$ exists & you have a simple pole with $L = c_{-1}$. I.e.,

then $$\lim_{z \to z_0} \left[f(z)(z-z_0)^n = \lim_{z \to z_0} \underbrace{f(z)(z-z_0)}_{L} \underbrace{(z-z_0)^{n-1}}_{0} \right] = L \cdot 0 = 0.$$

Simple poles are easy.

Ex: $f(z) = \dfrac{g(z)}{q(z)}$ has a simple pole at $z = z_0$ if $g(z)$ analytic in a nbhd of $z = z_0$, $g(z_0) \neq 0$, and q is a polynomial with simple root at $z = z_0$.

• f has a pole of order n at $z = z_0$ iff

$$f(z) = \frac{C_{-n}}{(z-z_0)^n} + \frac{C_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{C_{-1}}{z-z_0} + \sum_{k=0}^{\infty} C_k (z-z_0)^k$$

Thm ② f has a pole of order $n>0$ at $z=z_0$ iff $\lim_{z\to z_0} f(z)(z-z_0)^n = L \neq 0$. Then $L = C_{-n}$ and

$$\lim_{z\to z_0} f(z)(z-z_0)^m = \lim_{z\to z_0} \underbrace{f(z)(z-z_0)^n}_{C_{-n}} \underbrace{(z-z_0)^{m-n}}_{0} = 0 \quad \forall\, m>n.$$

Note: If $\lim_{z\to z_0} f(z)(z-z_0)^n = 0$ for $n = n_0 > 0$, then it holds for all $n > n_0$, and f must have a pole of order $< n_0$ at z_0.

Note: Thms ① & ② are clearly true from L-series, but L-series may not look at all like the given expression for f. Interestingly, you only need the L-series exists, but never need calculate it. ‼

Ex: $f(z) = \frac{g(z)}{q(z)}$ has a pole of order n at $z=z_0$ if g is analytic, q a polynomial with n'th order root at $z=z_0$, and $g(z_0) \neq 0$.

Q: what is the easiest way to calculate 4

C_{-1} at a pole of order n?

- Hard Way:

1. Find $C_{-n} = \lim_{z \to z_0} f(z)(z-z_0)^n$

   Subtract $\frac{C_{-n}}{(z-z_0)^n}$ to make $g_1(z) = f(z) - \frac{C_{-n}}{(z-z_0)^n}$

2. Find $C_{-(n-1)} = \underset{z \to z_0}{\text{limit}}\ g_1(z)(z-z_0)^{n-1}$

   Subtract $\frac{C_{-n+1}}{(z-z_0)^{n-1}}$ to make $g_2(z) = g_1(z) - \frac{C_{-n+1}}{(z-z_0)^{n-1}}$

$\vdots$ $(n-1)$-times

(n-1) Find $C_{-1} = \underset{z \to z_0}{\text{limit}}\ g_{n-1}(z)(z-z_0)$

That this works is immediate when you formally express the L-series for f, and perform the above operations.

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• **Easier Way**: Multiply $f(z)$ by power $(z-z_0)^n$ so that $f(z)(z-z_0)^n$ becomes a power series of positive powers of $(z-z_0)$. Then differentiate $n-1$ times & set $z=z_0$ to isolate c_{-1}. I.e., assuming $\lim_{z\to z_0} f(z)(z-z_0)^n = c_{-n} \neq 0$, so f has a pole of order n, the L-series is

$$f(z) = \frac{c_{-n}}{(z-z_0)^n} + \frac{c_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{c_{-1}}{z-z_0} + \sum_{k=0}^{\infty} c_k (z-z_0)^k$$

$$f(z)(z-z_0)^n = \underbrace{c_{-n} + c_{-n+1}(z-z_0) + \cdots +}_{\text{n-1 derivatives vanish}} \underbrace{c_{-1}(z-z_0)^{n-1}}_{\substack{\text{n-1 derivative} \\ \text{will isolate this term}}} + \sum_{k=0}^{\infty} c_k (z-z_0)^{k+n}$$

$$\frac{d^{n-1}}{dz^{n-1}}\left[f(z)(z-z_0)^n\right] = \frac{d^{n-1}}{dz^{n-1}}\left[c_{-1}(z-z_0)^{n-1} + \sum_{k=0}^{\infty} c_k (z-z_0)^{k+n}\right]$$

$$= (n-1)!\, c_{-1} + \underbrace{\frac{d^{n-1}}{dz^{n-1}}\left[\sum_{k=0}^{\infty} c_k (z-z)^{k+n}\right]}_{\text{This vanishes when you set } z=z_0}$$

Thus –

$$c_{-1} = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[f(z)(z-z_0)^n\right]\Bigg|_{z=z_0}$$

$$= \frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\left[f(z)(z-z_0)\right].$$

Thm 3 If f has a pole of order $n \geq 1$ at $z = z_0$, then

$$C_{-1} = \lim_{z \to z_0} \frac{g^{(n-1)}(z)}{(n-1)!} \equiv \frac{g^{(n-1)}(z_0)}{(n-1)!} = R(f, z_0)$$

where

$$g(z) = f(z)(z - z_0)^n.$$

Example 1: Evaluate: $I = \int_0^\infty \frac{x^2 dx}{(x^2+9)(x^2+4)^2}$.

Soln: By even symmetry, $I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+4)^2}$

Let

$$f(z) = \frac{z^2}{(z^2+9)(z^2+4)^2} = \frac{z^2}{(z-3i)(z+3i)(z-2i)^2(z+2i)^2}$$

f has simple poles: $z = \pm 3i$

double poles: $z = \pm 2i$

$$R(f, 3i) = \lim_{z \to 3i} f(z)(z-3i) = \frac{3i^2}{(3i+3i)(3i-2i)^2(3i+2i)^2}$$

$$= -\frac{3}{50i}$$

To find the residue at $z = 2i$ write

$$g(z) = (z-2i)^2 f(z) = \frac{z^2}{(z^2+9)(z+2i)^2}$$

$$R(f, 2i) = g'(2i)$$

$$= \frac{(-4+9)(2i+2i)^2(4i) - (2i)^2\left[5(2)(4i)+(4i)^2(4i)\right]}{5^2(4i)^2}$$

↓ Calc ↓

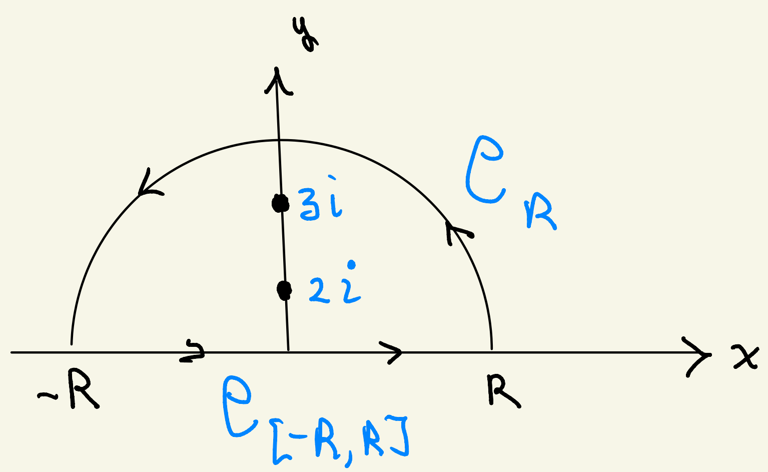
$$= -\frac{13i}{200}$$

Let

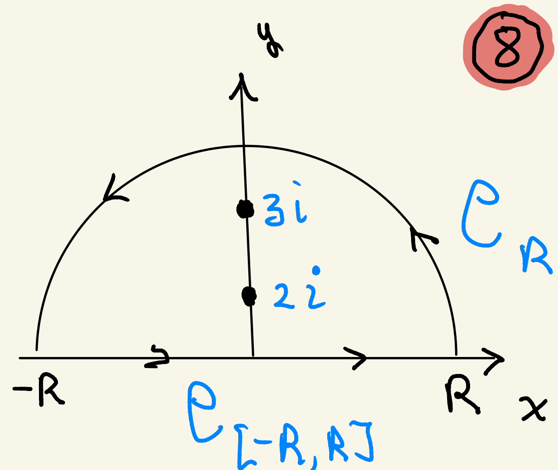
$C_R \equiv$ the upper $\frac{1}{2}$ circle of radius R, center $z=0$,

$C_{[-R,R]} \equiv$ closed interval $[-R,R] = \{x : -R \leq x \leq R\} \subseteq \mathbb{R}$

so $C = C_R + C_{[-R,R]}$ is a SCC containing singularities $z = 2i, 3i$:



Thus the Residue Thm implies



$$\oint_{C_R + C_{[-R,R]}} f(z)\,dz = 2\pi i \{R(f,2i) + R(f,3i)\}$$

$$= 2\pi i \left(-\frac{50}{50i} - \frac{13i}{200}\right) \underset{\text{Calc}}{=} \boxed{\pi/100}$$

But

$$\oint_{C_R + C_{[-R,R]}} f(z)\,dz = \underbrace{\int_{C_R} f(z)\,dz}_{\text{we show } \lim_{R\to\infty} = 0} + \underbrace{\int_{-R}^{R} f(x)\,dx}_{\lim_{R\to\infty} = \int_{-\infty}^{\infty} f(x)\,dx}$$

Estimate:

$$\left|\int_{C_R} f(z)\,dz\right| \leq \int_{C_R} |f(z)|\,|dz| \leq |C_R|\,M$$

$M = \operatorname{Max}_{C_R} |f(z)|$

$$|f(z)| = \left|\frac{z^2}{(z^2+9)(z^2+4)^2}\right| \leq \frac{|z|^2}{|z^2+9|\,|z^2+4|^2}$$

$$\operatorname{Max}_{C_R} |f(z)| \underset{R>9}{\leq} \frac{R^2}{(R^2-9)(R^2-4)^2} = \frac{R^2}{R^6\left(1-\frac{9}{R^2}\right)\left(1-\frac{4}{R^2}\right)^2}$$

$\big||a|-|b|\big| \leq |a+b| \leq |a|+|b|$

$$\leq \frac{1}{4} R^{-4}$$

for $\frac{9}{R^2} < \frac{1}{2}$, $\frac{4}{R^2} < \frac{1}{2}$

Thus:

$$\left|\int_{C_R} f(z)\,dz\right| \le \underbrace{|C_R|}_{\pi R}\ \underbrace{M}_{M \le \frac{1}{4}\frac{1}{R^4}} \le \frac{\pi}{4}\frac{1}{R^3} \xrightarrow[R\to\infty]{} 0$$

Conclude:

$$\underbrace{\frac{\pi}{100}}_{\text{Same value for every } R>0} = \underbrace{\int_{C_R} f(z)\,dz}_{\xrightarrow{R\to\infty} 0} + \underbrace{\int_{-R}^{R} f(x)\,dx}_{\xrightarrow{R\to\infty} I} \xrightarrow[R\to\infty]{} \int_{-\infty}^{\infty} f(x)\,dx = 2I$$

So

$$\boxed{I = \int_0^\infty \frac{x^2\,dx}{(x^2-9)(x^2-4)^2} = \frac{\pi}{200}}$$

The general case is developed in the next problem:

HW 9 Let $p(z), q(z)$ be polynomials such that $\deg(p)+2 \le \deg(q)$, and assume $p(x), q(x)$ are real.

Show: $\displaystyle\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx = 2\pi i \sum_{k=1}^{n} R\left(\frac{p(x)}{q(x)}, z_k\right)$, assuming $z_1, \dots, z_n$ are the roots of q enclosed by $C = C_R + C_{[-R,R]}$ for R sufficiently large, and assume no Real Roots.

Hint: Argue that $\displaystyle\lim_{R\to\infty} \int_{C_R} \frac{p(z)}{q(z)}\,dz = 0.$

Zeros of Analytic Functions:

Many of the functions we apply the Residue Theorem to look like

$$f(z) = \frac{g(z)}{h(z)}$$

where g and h are analytic. In this case the singularities in f are the zeros of h. If h is a polynomial of degree n, then f has singularities only at the n roots of h, and hence all singularities of f are isolated, and there are only a finite number of singularities inside any scc C. It turns out this conclusion holds for any analytic function $h(z)$.

Thm (Z) ("Zeros of an Analytic Function Theorem") The zeros of a non-constant analytic function $h(z)$ are always isolated, and the singularities in $f(z) = \frac{1}{h(z)}$ are always poles of finite order.

Holds for any f analytic in an open set $D \subseteq \mathbb{C}$.

(This is true fundamentally because analytic functions always admit a Taylor Series expansion - proof below.)

Theorem (Z) tells us that from the point of view of the Residue Thm, ratios of analytic functions "look like" polynomials.

Cor ① If g, h are analytic inside and in a nbhd of a scc $\mathcal{C}$, and h non-constant, then $f(z) = \frac{g(z)}{h(z)}$ has only a finite number of isolated singularities inside $\mathcal{C}$, and all of them are poles of finite order; I.e., $f(z) = \frac{g(z)}{h(z)}$ is always meromorphic inside $\mathcal{C}$. (Proof below)

Conclude: Essential Singularities in which $c_{-n} \neq 0$ for arbitrarily large n, never appear in the ratio of analytic functions!

The following generalizes HW 9

$c \equiv$ constant

HW 10 Assume $f(z), g(z)$ are entire functions such that $f(x), g(x)$ are real, and $\lim_{r\to\infty} \left|\frac{f(re^{i\theta})}{g(re^{i\theta})}\right| \leq \frac{c}{r^2}$.
Then $\int_{-\infty}^{\infty} \frac{f(x)}{g(x)}\,dx = 2\pi i \sum_{k=1}^{n} R\left(\frac{f}{g}, z_k\right)$ assuming $z_1, \ldots, z_n$ are the roots of g enclosed by $\mathcal{C} = \mathcal{C}_R + \mathcal{C}_{[-R,R]}$ for R sufficiently large, and $g(x) \neq 0$ for $x \in \mathbb{R}$.

Cor 2: If two analytic functions $f(z)$ & $g(z)$ agree on a convergent sequence of points $z_1, z_2, z_3 \ldots$ s.t. $z_n \longrightarrow \bar{z}$, then $f(z) = g(z)$ for all $z \in \mathbb{C}$.
(I.e., if f, g analytic in $D_{open} \subseteq \mathbb{C}$, $z_n, \bar{z} \in D$, then $f = g$ in D.)

In particular, this is a property of all real valued functions which extend to complex analytic functions, including real polynomials, together with all real valued functions which are expandable in power series!

☒ **Proof of Theorem (Z)**: Assume $g(z)$ is analytic, non-constant, and $g(z_0)=0$. Since g analytic, g can be expanded in T-series about $z=z_0$, $g(z)=\sum_{k=0}^{\infty} a_k (z-z_0)^k$. Since $g(z_0)=0$, setting $z=z_0 \Rightarrow a_0=0$. But g non-constant $\Rightarrow$ there exists smallest $n<\infty$ st $a_n \neq 0$. Factoring out $a_n(z-z_0)^n$ gives –

$$g(z) = \underbrace{a_n (z-z_0)^n}_{\text{isolated zero @ } z=z_0} \underbrace{\left\{1+\frac{a_{n+1}}{a_n}(z-z_0)+\frac{a_{n+2}}{a_n}(z-z_0^2)+\cdots\right\}}_{h(z)>0 \text{ analytic in nbhd of } z=z_0}.$$

Now $h(z_0)=1$, so $\frac{1}{h(z)}$ analytic in a nbhd of $z=z_0$;

so $\frac{1}{h(z)} = \sum_{k=0}^{\infty} \bar{a}_k (z-z_0)$, $\frac{1}{h(z_0)}=1 \Rightarrow \bar{a}_0=1.$

Thus

$$\frac{1}{g(z)} = \frac{a_n^{-1}}{(z-z_0)^n} \underbrace{\sum_{k=0}^{\infty} \bar{a}_k (z-z_0)^k}_{\text{T-series for } h^{-1}(z) \text{ in nbhd } z=z_0} \leftarrow \text{The Laurent exp. of } 1/g(z) \text{ about } z=z_0$$

<u>thus</u>: $\lim_{z\to z_0} \frac{1}{g(z)}\cdot(z-z_0)^n = a_n^{-1}\bar{a}_0 = C_{-1} \neq 0 \Rightarrow \frac{1}{g(z)}$ has pole order n at z_0. ✓

• Proof of Cor ①: Assume g, h are analytic functions inside and in nbhd of a scc C, $h \neq$ const. Let $D = \{z : z \text{ inside } C\}$ so D open, and $D \cup C = \bar{D}$. Since $\bar{D}$ is closed & bded, $\bar{D}$ is compact.

Now if $h(z) = 0$ for only a finite # of $z_k \in D$, $k = 1, \dots, n$ then $f = g/h$ has a finite # of poles inside C. ✓

Assume, then, for contradiction, that $\exists\ \infty$ sequence $\{z_k\}_{k=1}^{\infty}$ st $h(z_k) = 0$, and z_k inside C, so, $z_k \in D$. Since $\bar{D}$ is compact, there exists a convergent subsequence $z_{n_k} \to \bar{z} \in \bar{D}$. But h is analytic in an open nbhd of $\bar{D}$ $\Rightarrow$ h cont, so $\lim_{z_k \to \bar{z}} h(z_k) = h(\bar{z}) = 0$. Thus $\bar{z}$ is a non-isolated zero of h in an open set where h is analytic so Theorem (Z) $\Rightarrow$ $h =$ const ※ contradiction. Thus there are only a finite number of zeros of h, and hence singularities in g/h, inside C ✓

Proof of Cor ②: Assume f and g are analytic in some open set $D \subseteq \mathbb{C}$, and $f(z_n) = g(z_n)$ for $z_n \in D$, $z_n \longrightarrow \bar{z} \in D$. Then $f-g$ is analytic in D and $(f-g)(z_n) = 0$, $(f-g)(\bar{z}) = 0$. But Thm (Z) says that a non-zero analytic function can only have isolated zeroes in D, and $\bar{z} \in D$ is not isolated from other zeroes $z_n \in D$, because $z_n \longrightarrow \bar{z}$. I.e., every nbhd of $\bar{z}$ contains elements of $\{z_n\}$. Thus Thm Z implies $(f-g) \equiv 0$ in D, and hence

$$f(z) = g(z) \quad \forall \; z \in D.$$

This means f and g agree everywhere in D, as claimed. ✓

• **Example ②**: Consider Real integrals of type

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta, \qquad (Ex2)$$

where

$$F(\sin\theta, \cos\theta) = \frac{P(\sin\theta, \cos\theta)}{q(\sin\theta, \cos\theta)}$$

where $P(x,y)$ and $q(x,y)$ are polynomials.

The idea for evaluating integrals of type (Ex2) is to view (Ex2) as the parameterization of a complex line integral over $C \equiv$ unit circle.

I.e., let $z = e^{i\theta}$, $dz = i e^{i\theta} d\theta = i z\, d\theta$, $0 \le \theta \le 2\pi$.

Then $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \boxed{\dfrac{z - z^{-1}}{2i}}$, $\cos\theta = \boxed{\dfrac{z + z^{-1}}{2}}$

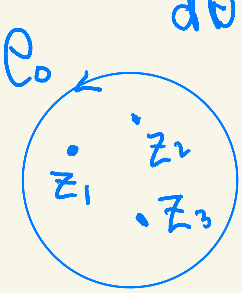
So (Ex2) is the parameterization of the complex line integral around unit circle

$$\int_0^{2\pi} F(\boxed{\sin\theta}, \boxed{\cos\theta})\, d\theta = \oint_{C_0} \underbrace{F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)(-i z^{-1})}_{f(z)}\, dz$$

$d\theta = -i z^{-1} dz$

$$= \oint_{C_0} f(z)\, dz$$

$$= 2\pi i \sum_{k=1}^{n} R(f, z_k)$$



$z_1, \dots, z_n$ the singularities of f inside C_0

Ex Evaluate $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{\frac{5}{4} + \sin\theta}$

Soln: $z = e^{i\theta} \Rightarrow d\theta = -i\frac{1}{z}dz$ and $\sin\theta = \dfrac{z - z^{-1}}{2i}$

So $$\frac{5}{4} + \sin\theta = \frac{5}{4} + \frac{z - \frac{1}{z}}{2i} = \frac{1}{4iz}(2z^2 + 5iz - 2)$$

Thus: $$I = \int_{\mathcal{C}_0} \frac{(4\cancel{i}\cancel{z})(\cancel{-}\cancel{i}\frac{1}{\cancel{z}})dz}{2z^2 + 5iz - 2} = \int_{\mathcal{C}_0} \frac{4\,dz}{(z+2i)(z+\frac{1}{2}i)}$$

Thus $I = 2\pi i R(f, -\frac{1}{2}i)$. We compute:

$$R(f, -\tfrac{1}{2}i) = \lim_{z \to \frac{1}{2}i} f(z)(z + \tfrac{1}{2}i)$$

$$= \frac{2}{-\frac{1}{2}i + 2i} = \frac{4}{3i}$$

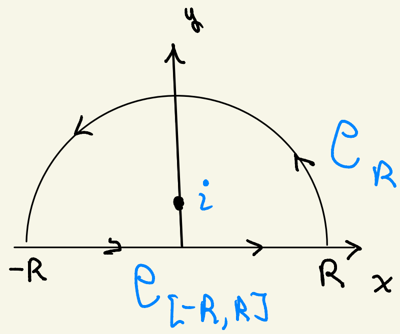
Conclude: $I = 2\pi\cancel{i}\,\dfrac{4}{3\cancel{i}} = \boxed{\dfrac{8\pi}{3}}$

• **Example ③** Improper integrals involving sin & cos: ⑱

Evaluate:

$$I = \int_0^\infty \frac{\cos x \, dx}{x^2+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x \, dx}{x^2+1}$$

Problem: If you naively try our trick of replacing x with z, integrate around $C = C_R + C_{[-R,R]}$ and take limit $R \to \infty$, you see quickly $\int_{C_R} f(z)dz$ does NOT tend to zero as $R \to \infty$.



I.e., $\cos z = \dfrac{e^{iz}+e^{-iz}}{2} = \dfrac{e^{-y}e^{ix}+e^{y}e^{-ix}}{2}$

and $e^{y} \longrightarrow \infty$ on C_R as $R \to \infty$!

Better Idea: note that $\cos x = \mathrm{Re}\{e^{ix}\}$.
Real Part

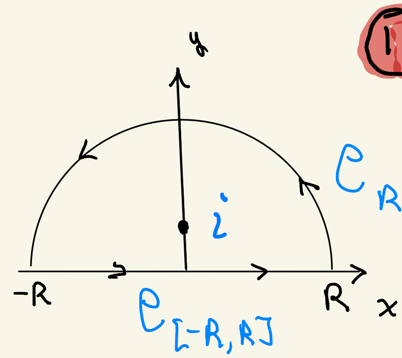
Thus $I = \frac{1}{2}\mathrm{Re}\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{x^2+1}$.

Now extend to complex values by defining

$$f(z) = \frac{e^{iz}}{z^2+1}.$$

Thus we try our trick with

$C = C_R + C_{[-R,R]}$ for the complex



function $f(z) = \dfrac{e^{iz}}{z^2+1}$, and extract

the real part at the end.

The trick requires $\displaystyle\int_{C_R} \frac{e^{iz}}{z^2+1}\,dz \xrightarrow[R\to 0]{} 0$.

Now $f(z) = \dfrac{e^{-y}e^{ix}}{(z-i)(z+i)}$, and C_R is in upper half

plane $y \geq 0$ so on C_R

$$|f(z)| \leq \frac{|e^{-y}||e^{ix}|}{|z^2+1|} \leq \frac{1}{R^2-1}$$

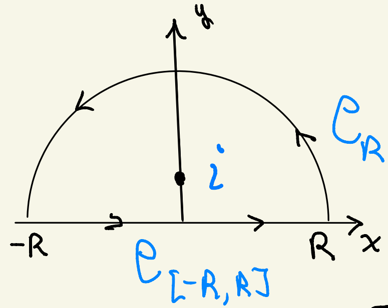
Thus

$$\left|\int_{C_R} f(z)\,dz\right| \leq |C_R|\,M \leq \pi R\,\frac{1}{R^2-1} \xrightarrow[R\to\infty]{} 0$$

$M = \max\limits_{z \in C_R} |f(z)|$

Now we can use our basic method:

$$\oint_{C_R + C_{[-R,R]}} f(z)\,dz = 2\pi i\,\mathrm{Res}(f, i) = \underbrace{\int_{C_R} f(z)\,dz}_{\xrightarrow{R\to\infty} 0} + \int_{[-R,R]} f(z)\,dz$$



Thus:

$$2\pi i\,\mathrm{Res}(f, i) = \lim_{R\to\infty}\left\{\oint_{C_R + C_{[-R,R]}} f(z)\,dz = \underbrace{\cancel{\int_{C_R}}}_{0} + \int_{C_{[-R,R]}}\right\}$$

$$= \lim_{R\to 0}\int_{-R}^{R}\frac{e^{ix}}{x^2+1}\,dx \equiv \int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+1}\,dx$$

$$\mathrm{Res}(f, i) = \lim_{z\to i} f(z)(z-i) = \lim_{z\to i}\frac{e^{iz}}{\cancel{(z-i)}(z+i)}\cancel{(z-i)}$$

$$= \frac{e^{i\cdot i}}{i+i} = \frac{e^{-1}}{2i} = \boxed{-\frac{i}{2e}}$$

$$2\pi i\,R(f, i) = 2\pi i\left(-\frac{i}{2e}\right) = \boxed{\pi/e}$$

Conclude:

$$I = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}\,dx = \frac{1}{2}\mathrm{Re}\left\{\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+1}\,dx\right\} = \frac{1}{2}\mathrm{Re}\left\{\frac{\pi}{e}\right\} = \boxed{\frac{\pi}{2e}}$$