V Topology of the Complex Plane: (Skip some proofs!)

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Defn: The "topology" of $\mathbb{C}$ or $\mathbb{R}^2$ is the collection of open sets of points in $\mathbb{R}^2$. The subject of topology is the characterization of convergence in terms of the open sets.

- Turns out - many notions related to convergence are better conceptualized when expressed in terms of open sets. The most important concept is compactness:
- The central problem of mathematical analysis is determining when an approximation scheme is actually converging - i.e., is it really approximating what it is supposed to approximate? For example, computers can only generate approximations to solutions of equations, so how do you determine the numerics is correct? The basic strategy of math analysis to prove the approximation sequence lies in a compact set, obtain a convergent subsequence, and prove its limit is an exact solution.

- Compactness is best expressed in terms of open sets, i.e. expressed "topologically".

Defn: A set $\mathcal{O} \subseteq \mathbb{C}$ (or $\mathbb{R}^2$) is open if

$$\forall z \in \mathcal{O} \ \exists \varepsilon > 0 \text{ st } B_\varepsilon(z) \subseteq \mathcal{O}.$$

"For every z in $\mathcal{O}$ there is a ball of radius $\varepsilon > 0$ centered at z which is also in $\mathcal{O}$"

A set $E \subseteq \mathbb{C}$ (or $\mathbb{R}^2$) is closed if $E = \mathcal{O}^c$ for some open set $\mathcal{O} \subseteq \mathbb{C}$. "Closed sets are the complement of open sets"

Here $\mathcal{O}^c$ = "$\mathcal{O}$ complement" $= \{z \in \mathbb{C} : z \notin \mathcal{O}\}$

Thm ①: ∞-unions and finite intersections of open sets are open.

∞-intersections and finite unions of closed sets are closed.

Proof De Morgan

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c \ ; \quad \left(\bigcap_\alpha E_\alpha\right)^c = \bigcup_\alpha E_\alpha^c$$

(HW) Check on $E_\alpha = (1+\alpha, 2] \subseteq \mathbb{R}$, $\alpha > 0$

Defn 1: A neighborhood (nbhd) of a point z_0 is any open set which contains z_0.

A deleted nbhd of z_0 is $\mathcal{O}_{open} \setminus \{z_0\}$ where $\mathcal{O}_{open}$ is an open set containing z_0.

(HW) Prove $\mathbb{C}$ and $\emptyset$ = empty set are the only sets which are both open and closed.

Defn 2: $\lim_{n\to\infty} z_n = z_0$ ($z_n \to z_0$) if $\forall \varepsilon > 0$ $\exists N$ st $n > N \Rightarrow |z_n - z_0| < \varepsilon$

Defn 3: $\lim_{z\to z_0} f(z) = L$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ st if $|z - z_0| < \delta$ and $z \neq z_0$ then $|f(z) - L| < \varepsilon$.

Since $|z_n - z_0| < \varepsilon \iff z_n \in B_\varepsilon(z_0)$ we say:

equivalent to or "iff"

"$z_n \to z_0$ if the sequence z_n is 'eventually' inside every nbhd of z_0."

meaning: "for n sufficien large"

"$f(z) \xrightarrow[z\to z_0]{} L$ if squeezing z within smaller and smaller deleted nbhds of z_0 forces $f(z)$ closer and closer to L."

(Note: Typically $f(z)$ not defined at z_0;

i.e. $\cancel{\lim_{\Delta z \to 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ not defined at $\Delta z = 0$.)

Thm ②: $\lim_{z \to z_0} f(z) = L$ iff for every sequence $z_n \to z_0$, $(z_n \neq z_0)$, we have $\lim_{n \to \infty} f(z_n) = L$.

Thm ③: If a limit of a sequence or function exists, then it is unique (there can be at most one limit)

Thm ④: $\lim_{n\to\infty} (\alpha z_n + \beta w_n) = \alpha \lim_{n\to\infty} z_n + \beta \lim_{n\to\infty} w_n$ $\quad \alpha, \beta \in \mathbb{C}$

$$\lim_{n\to\infty} z_n w_n = \left(\lim_{n\to\infty} z_n\right)\left(\lim_{n\to\infty} w_n\right)$$

$$\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{\lim_{n\to\infty} z_n}{\lim_{n\to\infty} w_n} \qquad \left(w_n, \lim_{n\to\infty} w_n \neq 0\right)$$

Thm ⑤: Same for limits of functions (HW)

Thm ⑥: (Big): A sequ. converges iff it is Cauchy

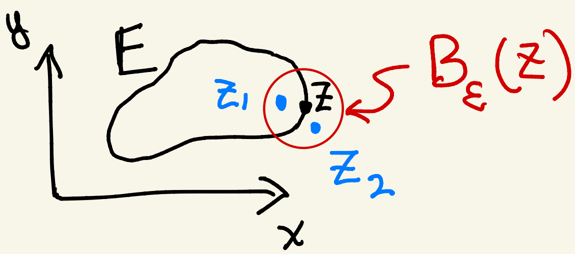
Defn ④: z_n is Cauchy if $\forall \varepsilon > 0 \ \exists N > 0$ st $m, n > N \Rightarrow |z_n - z_m| < \varepsilon$

implies

- Given a subset $E \subseteq \mathbb{C}$, the boundary of E, denoted ∂E, is the set of points $z \in \mathbb{C}$ such that every nbhd of z contains both points in E and in E^c. (Int $E = \{z \in E$ st $B_\varepsilon(z) \subseteq E$ some $\varepsilon > 0\}$)
Interior

- **Defn 5:** $z_0 \in \partial E$ if $\forall \varepsilon\ \exists\, z_1, z_2 \in \mathbb{C}$ st $z_1, z_2 \in B_\varepsilon(z_0)$ with $z_1 \in E$, $z_2 \notin E$ (i.e. $z_2 \in E^c$).

Picture: "z in boundary of E"



Turns Out: closed sets, defined as complements of open sets, are precisely the sets which contain their boundary!

Thm 6: $E \subseteq \mathbb{C}$ is closed iff $\partial E \subseteq E$

Defn 6: $\overline{E}$ = "closure of E" $= E \cup \partial E$

Thm 7: $\overline{E}$ is closed, and $\overline{\overline{E}} = \overline{E}$.

Cor: E is closed iff E is closed under limits — by which we mean that any point which is the limit of a sequence in E, is also in E.

- **Defn 7:** f is continuous at z_0 if $f(z_0)$ is defined, and $\lim_{z \to z_0} f(z) = z_0$.

The following famous theorem shows that continuity is a purely topological concept — i.e. can be characterized in terms of open sets —

Thm 8 (Big): f is continuous iff the inverse image $f^{-1}(\mathcal{O})$ of every open set $\mathcal{O} \subseteq \mathbb{C}$ is open.

Continuity is also expressed in terms of closed sets —

Thm 9: f continuous iff $f^{-1}(E)$ closed $\forall\, E$ closed.

Defn 8: $f^{-1}(E) = \{ z \in \mathbb{C} \text{ st } w = f(z) \text{ some } w \in E \}$.

Note: Thm's 8, 9 characterize functions $f: \mathbb{C} \to \mathbb{C}$ which are continuous at every $z \in \mathbb{C}$. If $f: D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ but not all of $\mathbb{C}$, then still true by defining $B \subseteq \mathbb{C}$ to be open relative to D if $B = D \cap \mathcal{O}_{open}$. Then f cont iff the pre-image of open sets are open relative to D.

- the most important concept in topology is compactness

Defn 9: $E \subseteq \mathbb{C}$ is compact if every open covering of E admits a finite subcover

That is, we say a collection of open sets $\{O_\alpha\}$ covers E if $E \subseteq \bigcup_\alpha O_\alpha$. E compact implies $E = O_{\alpha_1} \cup O_{\alpha_2} \cup \cdots \cup O_{\alpha_n}$ some finite subset of O_α's.

The following important theorem characterizes the compact subsets of $\mathbb{C}$ in terms of the topology alone.

Thm 10: (Big) A set $E \subseteq \mathbb{C}$ is compact iff it is closed and bounded.

Here, E is bounded if $\exists R>0$ st $E \subseteq B_R(0)$.

or $|z| \leq R$ $\forall z \in E$

Thm 11: (Big) $E \subseteq \mathbb{C}$ is closed & bounded (and hence compact) iff every sequence $z_n \in E$ has a convergent subsequence, and the limits of sequences in E, also lie in E

- Continuous functions defined on compact domains $D = E_{\text{compact}}$ admit mathematical analysis –

Thm 12 (Big): If a real valued function $f : E \to \mathbb{R}$ is continuous on compact set E, then:
(For complex fn's $f : E \to \mathbb{C}$ think $|f| : E \to \mathbb{R}$)

(1) f is bounded on E.
($\exists\, M > 0$ st $|f(z)| \leq M \;\forall\, z \in E$.)

(2) f takes on its max and min values on E
($\exists\, z_1, z_2 \in E$ st $|f(z_1)| \leq |f(z)| \leq |f(z_2)| \;\forall\, z \in E$)

Thm 13 (Big): A continuous function on a compact st E is uniformly continuous. (For us $f : E \to \mathbb{C}$ but this holds in general topological spaces)

Defn 10: $f : E \to \mathbb{C}$ is uniformly continuous if
$\forall\, \varepsilon > 0\; \exists\, \delta > 0$ st if $|z_2 - z_1| < \delta$ then $|f(z_2) - f(z_1)| < \varepsilon$.

"You can make outputs uniformly close by choosing any two inputs sufficiently close."

- The main problem of analysis is the problem of ensuring that approximation schemes are valid. This is the fundamental problem of computing – how do you know your numerical approximation is really approximating what you want?

Basic Problem – if you are approximating a function by a sequence of approximating functions $f_n \to f$, when can you infer continuity of limit f from continuity of the approximating f_n? Ans – need uniform limits!

Thm 14 (Big): If f_n are continuous and $f_n \to f$ uniformly on E, then f is continuous.

Defn 11: $f_n \to f$ uniformly on E (any E)

if $\forall\, \varepsilon > 0\ \exists\, N > 0$ st $n > N \underset{\text{implies}}{\Longrightarrow} |f_n(z) - f(z)| < \varepsilon$
$\forall\, z \in E$

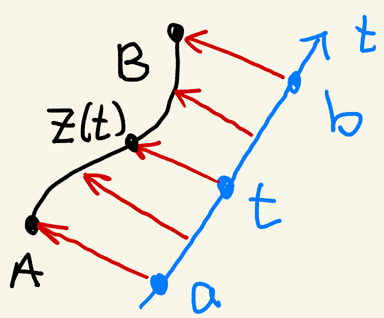
"You can make $f_n(z)$ uniformly close to $f(z)$ for all $z \in E$ by going sufficiently far out in your approximation sequence f_n"

Application: Uniform Convergence is the fundamental concept needed for the theory of Line Integrals —

- Let $\mathcal{C}$ be a curve in $\mathbb{C}$ defined by parameterization:

$$\mathcal{C}: \; z(t), \; a \le t \le b, \; z(a) = A, \; z(b) = B$$

Picture



Now assume $z(t)$ is continuous —

(HW) **Prove**: $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. if $|t_2 - t_1| < \delta$ then $|z(t_2) - z(t_1)| < \varepsilon$

Soln: $[a,b]$ closed & bounded $\Rightarrow$ compact. Since $z(t)$ continuous on compact $E = [a,b]$, Thm 13 implies $z(\cdot)$ is uniformly continuous. By Defn 10

$\forall \varepsilon > 0 \; \exists \delta > 0$ st $|t_2 - t_1| < \delta \Rightarrow |z(t_2) - z(t_1)| < \varepsilon$

as claimed ✓

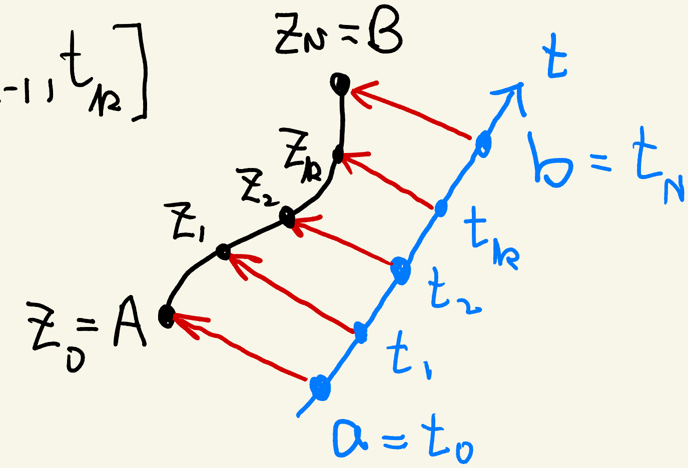
- Now create a Riemann Sum as follows:

Choose N large and define $\Delta t = \frac{b-a}{N}$ with

$$t_k = a + k\Delta t, \quad t_0 = a, \quad t_N = b$$

$$z_k = z(\bar{t}_k), \quad \bar{t}_k \in (t_{k-1}, t_k]$$

Picture:



(HW) Prove $|\Delta z_k| = |z_k - z_{k-1}|$ tends to zero uniformly as $N \to \infty$. I.e., $\forall \varepsilon > 0\ \exists \bar{N} > 0$ st

$$N > \bar{N} \Rightarrow |\Delta z_k| < \varepsilon \quad \forall\ k = 0, \dots, N.$$

If $z'(t)$ is also continuous, it follows that

$$|\Delta z_k'| = |z'(\bar{t}_k) - z'(\bar{t}_{k-1})| \longrightarrow 0 \text{ uniformly as } N \to \infty$$

as well.

- We now use the theory of uniform convergence to prove that when $f: \mathbb{C} \to \mathbb{C}$ is continuous, and $\mathcal{C}$ is a C^1-curve in the sense that $z(t), z'(t)$ both cont., then the line integral $\int_{\mathcal{C}} f(z)\,dz$ is a unique limit of Riemann Sums.

(HW) Thm (A): If $f: \mathbb{C} \to \mathbb{C}$ is continuous, and a curve $\mathcal{C}$ is C' in the sence that both $z(t)$ & $z'(t)$ are continuous for $a \le t \le b$, $z(a) = A$, $z(b) = b$, then

$$\int_{\mathcal{C}} f(z)dz = \int_a^b f(z(t))z'(t)dt = \lim_{N\to\infty} \sum_{k=1}^{N} f(z_k) z_k' \Delta t$$

Show this is equiv to $\int_{\mathcal{C}} \vec{G}_1 \cdot \vec{T} ds + i \int_{\mathcal{C}} \vec{G}_2 \cdot \vec{T} ds$

This is Riemann Sum with $z_k = z(\bar{t}_k)$, $z_k' = z'(\bar{t}_k)$ and $\Delta t = \frac{b-a}{N}$

is unique independent of how we choose $\bar{t}_k \in [t_{k-1}, t_k]$. (Assume one limit exists, cf. Math 127B)

Note: By the same argument we needn't choose t_k to be equally spaced, $t_k = k\Delta t$, $\Delta t = \frac{b-a}{N}$, we could take any $a = t_0 < \cdots < t_k < \cdots < t_N = b$ so long as $\max_{k=1}^{N} |t_k - t_{k-1}| \equiv \|\Delta t\| \xrightarrow[N\to\infty]{} 0$.

In this case we can take $z_k = z(\bar{t}_k)$, $z_k' = z'(\bar{t}_k)$ for any $\bar{t}_k \in [t_{k-1}, t_k]$.

Proof of Theorem (A): (Limits of Riemann Sums are unique)

Choose $a = t_0 < t_1 < \cdots < t_k < \cdots < t_N = b$, $\Delta t = \frac{b-a}{N}$, and set

$$z_k = z(\bar{t}_k),\ z_k' = z'(\bar{t}_k), \qquad t_{k-1} < \bar{t}_k \le t_k$$

and form the Riemann Sum

$$R_N = \sum_{k=1}^{N} f(z_k)\, z_k'\, \Delta t.$$

To verify Thm (A) (uniqueness), assume for a given choice of $\{\bar{t}_k\}_{k=1}^{N}$ at each value of N, the sequence of complex #'s $\{R_N\}$ converges as $N \to \infty$, i.e., $R_N \xrightarrow[N\to\infty]{} R_0 \in \mathbb{C}$. Then to get uniqueness, we must show that for any other choice of $\bar{t}_k^* \in (t_{k-1}, t_k]$ at each stage N, we get a different sequence

$$R_N' = \sum_{k=1}^{N} f(z_k^*)(z_k^*)'\, \Delta t,$$

and $R_N' \xrightarrow[N\to\infty]{} R_0$. So assume we know $\{R_N\}_{N=1}^{\infty}$ converges for one choice $\{\bar{t}_k\}$. (see Math 127B)

It suffices to show that $\forall \varepsilon > 0\ \exists \bar{N} > 0$ st $N > \bar{N} \overset{\text{implies}}{\Rightarrow} |R_N' - R_N| < \varepsilon$. This implies $R_N' \to R_0$ as well. (See argument at end.) For this, write

$$|R_N - R_N'| = \left| \sum_{k=0}^{N} f(z_k)\, z_k'\, \Delta t - \sum_{k=0}^{N} f(z_k^*)(z_k^*)'\, \Delta t \right|$$

$$= \left| \sum_{k=0}^{N} \left(f(z_k)\, z_k' - f(z_k^*)(z_k^*)' \right) \Delta t \right|$$

$$\leq \sum_{k=0}^{N} \left| f(z_k)\, z_k' - f(z_k^*)(z_k^*)' \right| \Delta t$$

Now the function $F(t) = f(z(t))\, z'(t)$ is cont on the compact interval $[a,b]$, so by Thm 13 F is uniformly continuous. Thus, $\forall \varepsilon > 0\ \exists \delta$ st if $|\Delta t| < \delta$, then $|F(t_k) - F(t_k^*)| < \varepsilon$.

...or we can make it $< \frac{\varepsilon}{|b-a|}$ if we like...
and we do like!

Since $\Delta t = \frac{b-a}{N}$, we can choose $\bar{N} = \frac{b-a}{\Delta t} >> 1$ large

so that $N > \bar{N} \Rightarrow \Delta t < \delta$ so that

$$\sum_{k=0}^{N} \underbrace{|F(\bar{t}_k) - F(\bar{t}_k^*)|}_{\leq \frac{\varepsilon}{|b-a|}} \Delta t = \sum_{k=0}^{N} \underbrace{|f(z_k) z_k' - f(z_k^*)(z_k')^*|}_{\leq \frac{\varepsilon}{|b-a|}} \Delta t$$

$$\leq \frac{\varepsilon}{|b-a|} \sum_{k=1}^{N} \Delta t = \frac{\varepsilon}{|b-a|} |b-a| = \varepsilon.$$

Conclude: $\forall\, \varepsilon > 0\ \exists\, \bar{N} > 0$ st $N > \bar{N} \Rightarrow |R_N - R_N'| < \varepsilon$.

It follows that if $R_N \to R_0$, then also $R_N' \to R_0$.

(I.e., $|R_N' - R_0| = |R_N' - R_N + R_N - R_0| \leq |R_N' - R_N| + |R_N - R_0|$

so given $\varepsilon > 0$, choose $\bar{N}$ st $N > \bar{N} \Rightarrow |R_N' - R_N| < \frac{\varepsilon}{2}$ & $|R_N - R_0| < \frac{\varepsilon}{2}$

$\Rightarrow |R_N' - R_0| < \varepsilon$ ✓)

(HW) Thm (B): Prove that the length of a curve $L = \int_a^b |z'(t)|\,dt$ defined by

$$L = \int_a^b |z'(t)|\,dt = \lim_{N\to\infty} \sum_{k=1}^{N} |z(t_k^*) - z(t_{k-1}^*)|$$

is defined independently of how we choose $t_k^* \in (t_{k-1}, t_k]$. (This uses essentially the same argument as Thm (A), and generalizes to indept of mesh $\{t_k\}_{k=1}^{N}$, so long as $\|\Delta t\| \xrightarrow[N\to\infty]{} 0$.)

(HW) Thm (C) Prove

$$\left| \int_a^b f(z(t))\, z'(t)\,dt \right| \le LM$$

where $M = \max_{a \le t \le b} |f(z(t))|$.