VI Cauchy-Goursat Theorem

- Essentially, the C-G Theorem gives an integral representation of analytic functions which tells us that there are no low regularity analytic functions. I.e., if $f : D \to \mathbb{C}$ has a derivative in an open set D, then $f^{(n)}(z)$ exists $\forall z \in D$. We then use C-G to prove that the Taylor Series for f converges in a nbhd of every point $z_0 \in D$; ie

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n , \quad a_n = \frac{f^{(n)}(z_0)}{n!}$$

converges for $|z - z_0| < R$, R = largest radius such that $B_R(z_0) \subseteq D$.

(HW) Not every C^∞ real f is "analytic".

Ex:

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Show f has no complex extension. Explain.

- What we have so far — (We be careful about assumptions)

- Defn: $f: D \to \mathbb{C}$ is analytic in an open set D if, for every $z \in D$,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

exists indept of how $\Delta z \to 0$.

- Our starting point was

Theorem (Cauchy-Riemann): Assume $u(x,y)$ and $v(x,y)$ are differentiable at $\overline{(x_0, y_0)}$. Then $f'(z_0)$ exists at $z_0 = x_0 + i y_0$ iff the Cauchy-Riemann Equations (CR) hold:

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x . \end{aligned} \qquad \text{(CR)}$$

(*) Note: At this stage we do not need assume derivatives u_x, u_y, v_x, v_y are continuous, or even exist away from z_0.

- Recall: It was not enough for (CR) alone to hold at $z = z_0$ to conclude $f'(z_0)$ exists – we needed u, v be differentiable at z_0 as well – CR and differentiability are pointwise conditions
- Recall: u_x and u_y could exist at z_0 and u not differentiable – for differentiable we need that all directional derivatives exist at z_0, i.e., $\nabla u \cdot \vec{v} = \frac{d}{dt} u(\underset{\sim}{x}_0 + t\vec{v})$ exists for all $\vec{v}$, which is equivalent to the existence of a tangent plane to u at $\underset{\sim}{x}_0$
- Theorem (Math 127A) If u_x and u_y are continuous in D, then u is continuously differentiable in D, so $f'(z)$ is continuous
- Our Question: Do there exist "low regularity" analytic functions, by which we mean $f'(z)$ exists $\forall\, z \in D$, but $f'(z)$ is not continuous?

Ans: NO!

HW Show that $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & 0 \end{cases}$ is a real valued function whose derivative $f'(x)$ exists for every $x \in \mathbb{R}$, but is discontinuous at $x=0$.

(This is a subtle issue because $f'(x)$ will not have simple jump discontinuities when f' is not continuous!)

By (C-G) we prove there are no such examples for complex differentiable functions.

- We have established that $f: D \to \mathbb{C}$ with u, v differentiable, satisfy CR in D iff $f'(z)$ exists (is analytic), but this does not rule out $f'(z)$ discontinuous. Our development of the FTC did require $f'(z)$ continuous to apply our theory of Line Integrals.
- I.e., our extension of the FTC was based on Green's Theorem, and this required not only that f be analytic, but also required $f'(z)$ be continuous. This is because we use the theorem that the Riemann Integral exists (the Riemann Sums defining it converge) IF the integrated function is continuous - and in Green's Thm $\iint_{A} N_x - M_y \, dA = \int_{C} \vec{F} \cdot \vec{T} \, ds$, N_x & N_y are integrated.

- Our extension of FTC to complex $f: D \to \mathbb{C}$ we've established so far can be stated precisely as:

Theorem (FTC): If $f: D \to \mathbb{C}$ is analytic ($f'(z)$ exists) in an open set $D \subseteq \mathbb{C}$, such that D is simply connected and $f'(z)$ is continuous, then

(1) f has an anti-derivative F st $F'(z) = f(z)$

(2) The FTC holds: $\int_C f(z)\,dz = F(B) - F(A)$

(3) $\int_C f(z)\,dz = 0$

Thus: If we can prove that whenever $f'(z)$ exists in D, we also know $f'(z)$ continuous, Theorem (FTC) extends to every analytic function. We get this from Cauchy-Goursat.

- **Aside**: Recall where in proof of FTC we required $f'(z)$ continuous:

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy) = \int_C \vec{G}_1\cdot\vec{T}\,ds + i\int_C \vec{G}_2\cdot\vec{T}\,ds$$

$\vec{G}_1 = (u,-v)$ $\quad \vec{G}_2 = \overrightarrow{(v,u)}$

We then showed (CR) $\Rightarrow \vec{G}_1$ & $\vec{G}_2$ Curl free,

$$\text{Curl}\,\vec{G}_1 = 0 = \text{Curl}\,\vec{G}_2 \quad \text{when } u_x = v_y,\ u_y = -v_x$$

then Green's Thm $\Rightarrow \vec{G}_1$ and $\vec{G}_2$ conservative, i.e.

$$0 = \iint_A \text{Curl}\,\vec{G}_i\cdot\vec{n}\,dA \underset{\uparrow}{=} \oint_C \vec{G}_i\cdot\vec{T}\,ds \underset{\uparrow}{=} 0$$

Curl-free $\vec{G}_i$ $\quad$ Gn's Thm $\quad$ every closed curve $C \Rightarrow$ conservative $\Rightarrow \vec{G}_1 = \nabla U,\ \vec{G}_2 = \nabla V$

But $\iint_A \text{Curl}\,\vec{G}\cdot n\,dA = \iint_A N_x - M_y\,dA$ for $\vec{G} = \overrightarrow{(M,N)}$,

and our theory of the Riemann Integral requires the integrand, $\text{Curl}\,\vec{G}\cdot\vec{n} = N_x - M_y$, be continuous!

Conclude: Our Complex FTC only holds for analytic functions whose derivative $f'(z)$ is continuous. Our **Goal** now is to prove

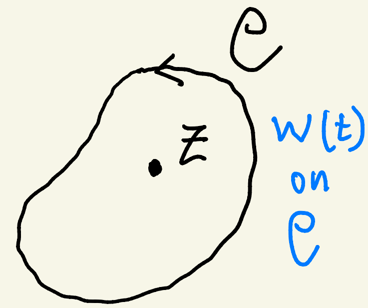
$f: D \to \mathbb{C}$ analytic (f' exists) $\Rightarrow$ f' continuous $\Rightarrow f^{(n)}$ cont.

The Cauchy-Goursat Theorem

Theorem (C-G): If $f: D \to \mathbb{C}$ is analytic ($f'(z)$ exists) in a simply connected open set $D \subseteq \mathbb{C}$, then

$$\oint_{\mathcal{C}} f(z)\,dz = 0$$

for every closed curve $\mathcal{C}$ in D.



Corollary (Cauchy Integral Formula)

Assume f is analytic in a S.C. domain $D_{\text{open}} \subseteq \mathbb{C}$.

Then

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{w-z}\,dw. \qquad \text{(CIF)}$$

((CIF) holds for any positively oriented simple closed curve in D that is C^1 and contains z.)

Main Point: Neither requires $f'(z)$ be continuous, only that $f'(z)$ exists. Because the integrand $\dfrac{f(w)}{w-z}$ is a continuous function of w on $\mathcal{C}$, we can differentiate thru the integral sign and get a formula for all derivatives of $f(z)$!

That is: $2\pi i f(z) = \oint_{\mathcal{C}} \frac{f(w)}{w-z}\,dw \Rightarrow$

$$2\pi i f'(z) = \frac{d}{dz}\oint_{\mathcal{C}} \frac{f(w)}{w-z}\,dw = \oint_{\mathcal{C}} \frac{\partial}{\partial z}\frac{f(w)}{w-z}\,dw = \oint_{\mathcal{C}} \frac{f(w)}{(w-z)^2}\,dw$$

justified if $f'(w)$ cont. on $\mathcal{C}$

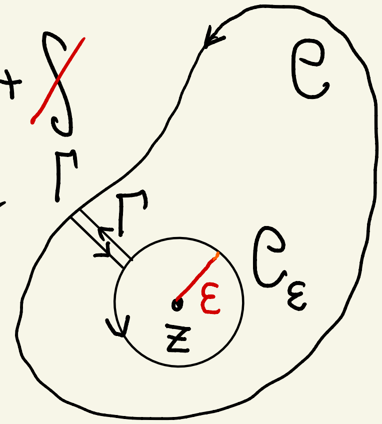
continuous fn of $z \notin \mathcal{C}$!

- We first prove (CIF) assuming (C-G), and then return to the proof of (C-G), which is a landmark proof in history of mathematics!

Proof of (CIF): Assume the (C-G) Theorem, which states that if f is analytic in s.c. domain D, then $\oint_{\mathcal{C}} f(z)dz = 0$ $\forall$ closed curve $\mathcal{C}$. Let $f: D \to \mathbb{C}$ be analytic. We show (CIF) holds. But f analytic $\Rightarrow$ f continuous where $w \neq z$, so cont in w along any $\mathcal{C}$ that avoids $w=z$. If $\mathcal{C}$ does not wind around z, then f is s.c. inside $\mathcal{C}$, hence (C-G) $\Rightarrow$ $\oint_{\mathcal{C}} \frac{f(w)}{w-z}\,dw = 0$, $\forall$ closed curve in D not containing z.

Now let C_ε denote the positively oriented circle of radius ε, center $z \in D$. Then

$$0 = \oint_{\underbrace{C - \Gamma - C_\varepsilon + \Gamma}_{\text{a closed curve not containing } z}} \frac{f(w)}{w-z}\,dw = \oint_{C} \frac{f(w)}{w-z}\,dw - \cancel{\int_{\Gamma}} - \oint_{C_\varepsilon} + \cancel{\int_{\Gamma}}$$



So

$$\oint_{C} \frac{f(w)}{w-z}\,dw = \oint_{C_\varepsilon} \frac{f(w)}{w-z}\,dw.$$

Now since $f(w)$ is continuous on C_ε, as $\varepsilon \to 0$, $f(w)$ tends to its value $f(z)$ at the center of C_ε. Thus by continuity and properties of integrals we have –

$$\oint_{C_\varepsilon} \frac{f(w)}{w-z}\,dw = f(z) \underbrace{\oint_{C_\varepsilon} \frac{dw}{w-z}}_{= 2\pi i} + o(1)$$

"little oh of one" tends to zero as $\varepsilon \to 0$

Thus $\displaystyle\oint_{C} \frac{f(w)}{w-z}\,dw = 2\pi i\, f(z) + o(\varepsilon)$. Taking $\varepsilon \to 0$,

$$o(\varepsilon) \to 0 \implies \boxed{f(z) = \frac{1}{2\pi i} \oint_{C} \frac{f(w)}{w-z}\,dw}$$ which is (CIF) ✓

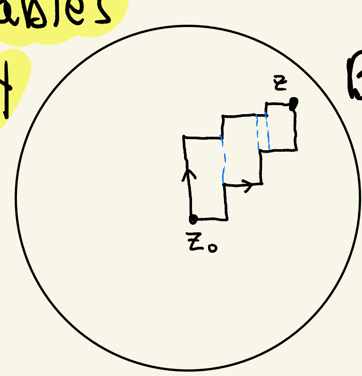
Proof of the Cauchy-Goursat Thm: (famous!)

- We do the case $D = B_r(z_0) \subseteq \mathbb{C}$. This contains all the ideas, and extension to any simply connected domain can be done by deforming a general curve into such a D.
- There are two main parts to the Proof:

(I) Main Lemma: If C is a rectangle $R \subseteq D$, then $\oint_C f(z)\,dz = 0$.

(II) Using (I) we construct an anti-derivative $F(z) = \int_C f(w)\,dw = \int_{z_0}^{z} f(w)\,dw$ where C is a curve which follows the sides of rectangles in D. Once we get $F'(z) = f(z)$, we have FTC so $\oint_C f(z)\,dz = F(B) - F(A) = 0$.
$A = B$

The key is Part (I), which enables us to get a path independent integral using simple straight line paths...



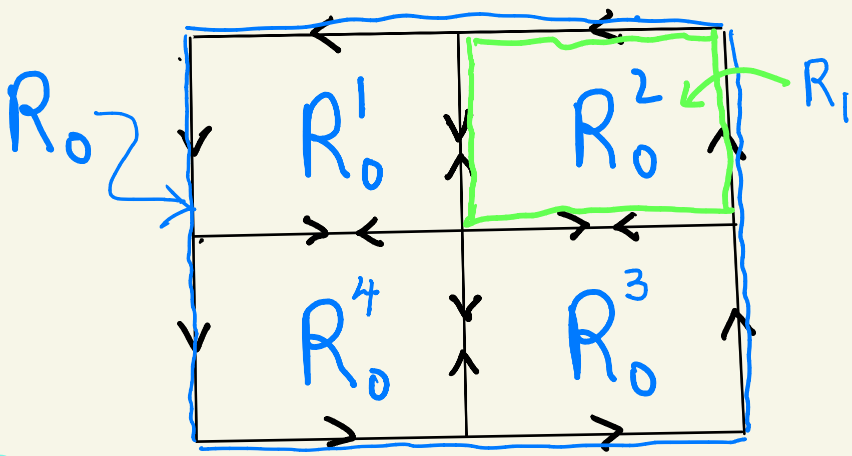
$B_r(z_0) = D$

◆ **Proof Part (I)** for the case $D = B_r(z_0) \subseteq \mathbb{C}$.

So assume $f: D \to \mathbb{C}$ is analytic ($f'(z)$ exists) in $B_r(z_0)$.

Main Lemma: Assume C_{R_0} is the pos. oriented boundary of rectangle $R_0 \subseteq B_r(z_0)$. Then $\int_{C_{R_0}} f(z)\,dz = 0$.

- Divide the rectangle R_0 into 4 equal subrectangles $R_0^1, R_0^2, R_0^3, R_0^4$, (labeled say clockwise), and write $C_{R_0} = C_{R_0^1} + C_{R_0^2} + C_{R_0^3} + C_{R_0^4}$



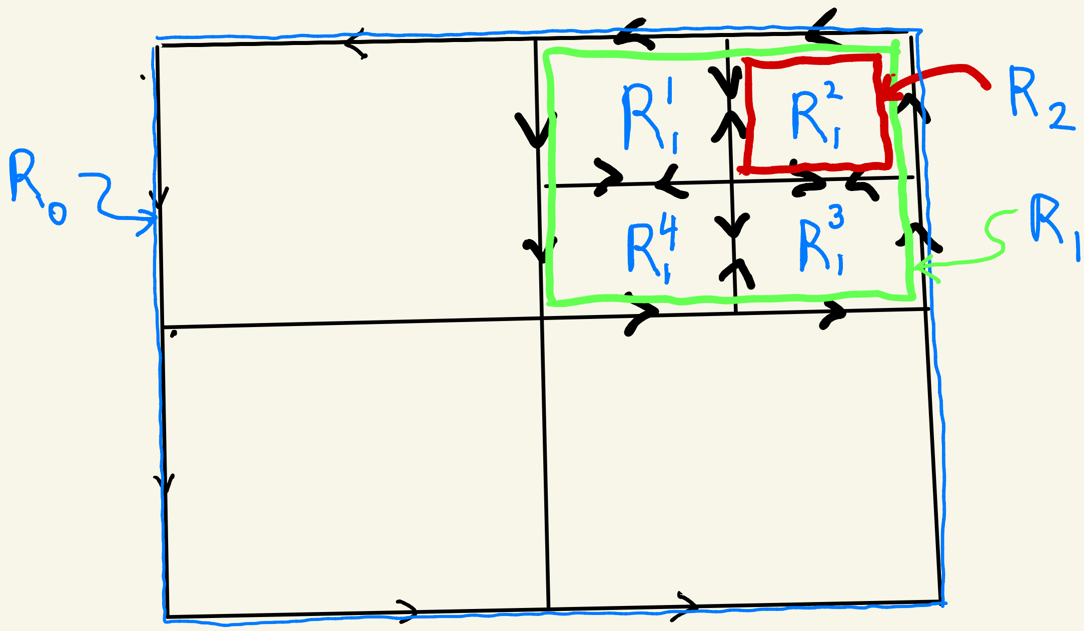
- Estimate:

$$\left|\oint_{C_{R_0}} f(z)\,dz\right| \le \sum_{k=1}^{4} \left|\oint_{C_{R_0^k}} f(z)\,dz\right| \le 4\left|\oint_{C_{R_1}} f(z)\,dz\right|$$

I.e., Choose R_1 so $\left|\oint_{C_{R_1}} f(z)\,dz\right| = \max_{k=1,2,3,4} \left|\oint_{C_{R_0^k}} f(z)\,dz\right|$

- **Next:** partition R_1 into 4 equal rectangles

$R_1 = R_1^1, R_1^2, R_1^3, R_1^4,$



and choose R_2 to be R_1^k such that

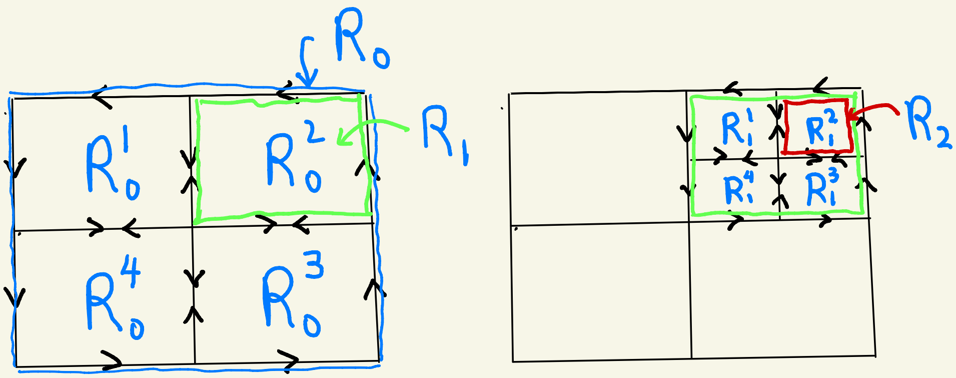
$$\left| \oint_{\mathcal{C}_{R_2}} f(z)\,dz \right| = \max_{k=1,2,3,4} \left| \oint_{\mathcal{C}_{R_1^k}} f(z)\,dz \right|$$

so that

$$\left| \oint_{\mathcal{C}_{R_1}} f(z)\,dz \right| \leq 4 \left| \oint_{\mathcal{C}_{R_2}} f(z)\,dz \right|$$

Hence:

$$\left|\oint_{\mathcal{C}_{R_0}} f(z)\,dz\right| \le 4 \left|\oint_{\mathcal{C}_{R_1}} f(z)\,dz\right| \le (4)^2 \left|\oint_{\mathcal{C}_{R_2}} f(z)\,dz\right|$$

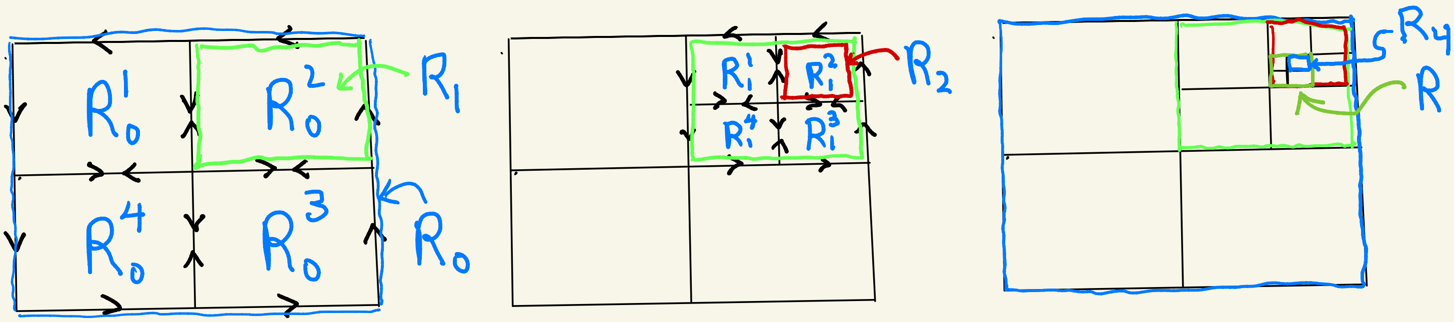


- Continue to get a sequence of nested rectangles

$$R_0 \supseteq R_1 \supseteq R_2 \supseteq \cdots \supseteq R_n \supseteq \cdots$$

such that

$$\left|\oint_{\mathcal{C}_{R_0}} f(z)\,dz\right| \le 4^n \left|\oint_{\mathcal{C}_{R_n}} f(z)\,dz\right|, \quad n = 1, 2, 3, \dots$$



$$R_0 \supseteq R_1 \supseteq R_2 \supseteq R_3 \supseteq R_4 \supseteq \cdots$$

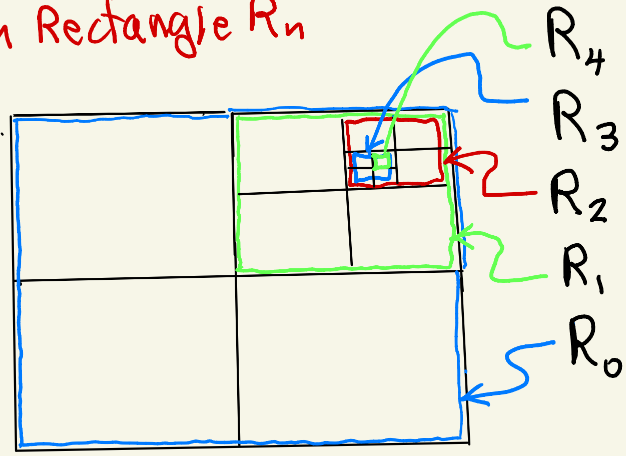
- So Consider the estimate

$$\left| \oint_{C_{R_0}} f(z)\,dz \right| \leq 4^n \left| \oint_{C_{R_n}} f(z)\,dz \right| , \quad n=1,2,3\ldots \qquad (*)$$

Looking to apply estimate

$$\left| \oint_{C_{R_n}} f(z)\,dz \right| \leq \underbrace{\max_{z\in R_n} |f(z)|}_{\text{max modulus } |f(z)| \text{ in n'th Rectangle } R_n} \; \underbrace{|C_{R_n}|}_{\text{perimiter of } R_n}$$

we estimate $|C_{R_n}|$, which for simplicity we denote $|C_{R_n}| = |R_n|$.



- Let $|R_0|$ = length of perimeter of R_0 $= |C_{R_0}|$.

Then $|R_1| = \frac{1}{2}|R_0|$, $|R_2| = \frac{1}{2}|R_1|, \ldots \Longrightarrow$ implies

$$\boxed{|R_n| = \frac{1}{2}|R_{n-1}|}$$

So

$$|R_n| = \frac{1}{2}|R_{n-1}| = \frac{1}{4}|R_{n-2}| = \cdots \frac{1}{2^n}|R_0|$$

$$\boxed{|R_n| = \frac{1}{2^n}|R_0|}$$

Conclude: $$|C_{R_n}| \leq \frac{1}{2^n}|C_{R_0}| = \frac{1}{2^n}\underbrace{|R_0|}_{\text{length of perimeter of the original rectangle}} \qquad (**)$$

- Consider now the nested sequence of rectangles:

$$R_0 \supset R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots$$

<u>Claim:</u> $\bigcap_{n=0}^{\infty} R_n$ = a single point $(\bar{x}, \bar{y}) \in R_0$

To see this note that $R_n = [a_n, b_n] \times [c_n, d_n]$ where $\{[a_n, b_n]\}_{n=0}^{\infty}$ and $\{[c_n, d_n]\}_{n=0}^{\infty}$ are each a nested sequence of intervals,

$[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$ and $[c_n, d_n] \subseteq [c_{n-1}, d_{n-1}]$

$n = 0, 1, 2 \ldots$ $\qquad$ $n = 0, 1, 2, \ldots$

and $|b_n - a_n| \xrightarrow[n\to\infty]{} 0$, $|d_n - c_n| \xrightarrow[n\to\infty]{} 0$.

Thus by the Nested Intervals Theorem, (Math 127A)

$\bigcap_{n=0}^{\infty} [a_n, b_n] = \{\bar{x}\}$ and $\bigcap_{n=0}^{\infty} [c_n, d_n] = \{\bar{y}\}$,

and hence $\bigcap_{n=0}^{\infty} R_n$ = a single point $(\bar{x}, \bar{y}) \in R_0$ ✓

- Since we assume $f(z)$ differentiable in $D \supseteq R_0$, f is differentiable at $\bar{z} = \bar{x} + i\bar{y}$, and so

$$\lim_{\Delta z \to 0} \frac{f(z) - f(\bar{z})}{\Delta z} = f'(\bar{z}),$$

So

$$f(z) = f(\bar{z}) + f'(\bar{z})\Delta z + o(1)\Delta z$$

little 'oh' of 'one' tends to zero as $n \to \infty$ $z \to \bar{z}$

Assuming $\Delta z = z - \bar{z}$ for $z \in R_n$, (we know $\bar{z} \in R_n \ \forall n$)

$$\left| \oint_{\mathcal{C}_{R_n}} f(z)\,dz \right| = \left| \oint_{\mathcal{C}_{R_n}} f(\bar{z}) + f'(\bar{z})(z - \bar{z}) + o(1)\Delta z \right|$$

linear analytic function $\Rightarrow \int_{\mathcal{C}_{R_n}} f(\bar{z}) + f'(\bar{z})(z - \bar{z})\,dz = 0$

$$= \left| \oint_{\mathcal{C}_{R_n}} o(1)\Delta z\,dz \right| \leq \oint_{\mathcal{C}_{R_n}} o(1)\,|\Delta z|\,dz$$

$|z - \bar{z}| \leq |R_n| \leq \frac{1}{2^n}|R_0|$

$$\leq |\mathcal{C}_{R_n}|\, o(1) \frac{1}{2^n} |R_0| \leq o(1)\left(\frac{1}{4}\right)^n |R_0|^2$$

by (**)

Conclude:

$$\boxed{\left| \int_{\mathcal{C}_{R_n}} f(z)\,dz \right| \leq o(1)\left(\frac{1}{4}\right)^n |R_0|^2} \qquad (***)$$

- Putting (***) into (*) gives:

$$\left|\oint_{\mathcal{C}_{R_0}} f(z)\,dz\right| \le \cancel{4^n}\,\frac{|R_0|^2}{\cancel{4^n}}\,o(1) = o(1)\,|R_0|^2 \xrightarrow[n\to\infty]{} 0$$

Conclude: The only way this holds $\forall n$ is if

$$\boxed{\oint_{\mathcal{C}_{R_0}} f(z)\,dz = 0}$$

as claimed ✓

This completes the proof of (I) Main Lemma – namely, assuming only that $f'(z)$ exists for $z \in D = B_r(z_0)$, it follows that the integral around every closed rectangle R_0 in D, equals zero!

- Aside: the estimate $\left|\oint_{\mathcal{C}_{R_n}} f(z)\,dz\right| \le \operatorname{Max}_{z\in R_n} |f(z)|\,|\mathcal{C}_{R_n}|$ is too "cheap" because $\operatorname{Max}_{z\in R_n} |f(z)| \ge C > 0$ together with $|\mathcal{C}_{R_n}| = |R_n| = \frac{1}{2^n}|R_0|$ only gives

$$\oint_{\mathcal{C}_{R_0}} f(z)\,dz \le 4^n \oint_{\mathcal{C}_{R_n}} f(z)\,dz \le C\,4^n\,\frac{1}{2^n}\,|R_0| \not\to 0$$

so we need to use f analytic!

$f'(z)$ gets us 2 orders better, $|f| \le C \to o(1)\,\Delta z$

(II) Using (I) Main Lemma we construct an anti-derivative of $f(z)$ using integration along sides of rectangles with sides $\parallel$ x & y-axes.

- To start, lets recall how it is that if we know $f: D \to \mathbb{C}$ is analytic in D_{open} (so $f'(z)$ exists but we don't assume $f'(z)$ continuous), then the existence of anti-derivative $F'(z) = f(z)$ by itself implies $\oint_{\mathcal{C}} f(z)dz = 0$ for every closed curve in D.

[Recall we needed $f'(z)$ cont to construct $F(z)$ from
(CR) $\Rightarrow \operatorname{Curl}\vec{G}_1 = 0 = \operatorname{Curl}\vec{G}_2 \Rightarrow \vec{G}_1 = \nabla U, \vec{G}_2 = \nabla V \Rightarrow F(z) = U(z) + iV(z)$)
Need $f'(z)$ continuous to apply Green's Thm!]

I.e., $F(z) = U(z) + iV(z)$ and $F'(z) = f(z) = u + iv \Rightarrow$

$(\Delta z = \Delta x)$ $U_x = u \underset{CR}{=} V_y$ & $V_x = v \underset{CR}{=} -U_y \Rightarrow$

$\nabla U = (U_x, U_y) = \overrightarrow{(u, -v)} = \vec{G}_1$, $\nabla V = \overrightarrow{(v, u)} = \vec{G}_2 \Rightarrow$

$$\int_{\mathcal{C}} f(z)dz = \int_{\mathcal{C}} \vec{G}_1 \cdot \vec{T} ds + i \int_{\mathcal{C}} \vec{G}_2 \cdot \vec{T} ds = \int_{\mathcal{C}} \nabla U \cdot \vec{T} ds + i \int_{\mathcal{C}} \nabla V \cdot \vec{T} ds$$

$$= U(B) - U(A) + iV(B) - iV(A)$$

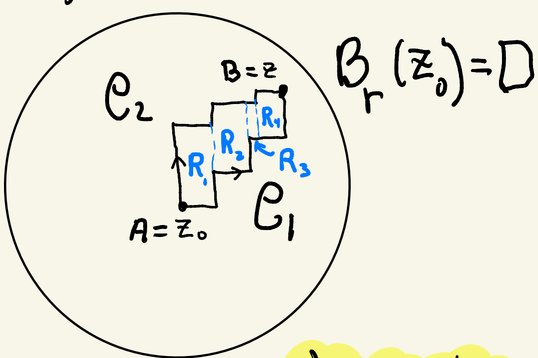
$$= F(B) - F(A) \checkmark$$

$\Rightarrow \oint_{\mathcal{C}} f(z)dz = 0 \checkmark$

$A = B$

- Thus it suffices to construct F such that $F'(z) = f(z)$ to prove $\oint f(z)\,dz = 0$ $\forall$ closed $\mathcal{C}$ in D. For simplicity, we do case $D = B_r(z_0)$.
- Because $\oint_{\mathcal{C}_R} f(z)\,dz = 0$ for every closed rectangle $R \subseteq B_r(z_0)$, it follows that $\int_{\mathcal{C}} f(z)\,dz$ is the same for any "rectangular curve" following the sides of rectangles parallel to x,y-axes, as pictured.

That is, $\int_{\mathcal{C}_1} f(z)\,dz = \int_{\mathcal{C}_2} f(z)\,dz$



because the space between $\mathcal{C}_1$ and $\mathcal{C}_2$ can be filled with closed rectangles, say $R_1 \cup R_2 \cup R_3 \cup R_4$ as in Figure, and so

$$0 = \int_{\mathcal{C}_{R_1}+\mathcal{C}_{R_2}+\mathcal{C}_{R_3}+\mathcal{C}_{R_4}} f(z)\,dz = \int_{\mathcal{C}_2} f(z)\,dz - \int_{\mathcal{C}_1} f(z)\,dz$$

integrals along shared sides cancel!

So $\int_{\mathcal{C}_2} f(z)\,dz = \int_{\mathcal{C}_1} f(z)\,dz$ is indept of rectangular path taking $A \to B$.

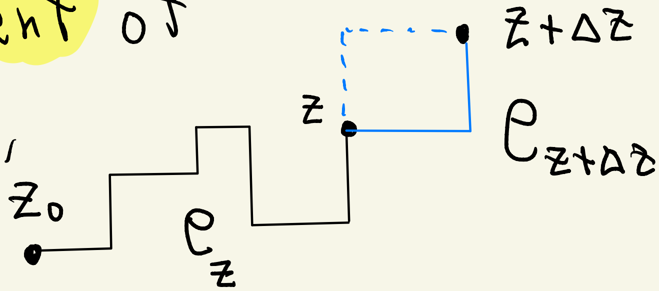
• **Define** $F(z) = \int_{C_z} f(z)\,dz$ for any $z \in B_r(z_0)$,

where the integral can be taken along any rectangular path C_z taking $A = z_0$ and $B = z$.

Claim: $F'(z) = f(z)$. For this consider

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\left\{ \int_{C_{z+\Delta z}} f(w)\,dw - \int_{C_z} f(w)\,dw \right\}$$

Since integrals are independent of rectangular path chosen, assume $C_{z+\Delta z}$ is the rectangular path C_z from z_0 to z, and the sides of the single rectangle from z to $z+\Delta z$.



Then $\displaystyle\int_{C_{z+\Delta z}} f(w)\,dw - \int_{C_z} f(w)\,dw = \int_z^{z+\Delta z} f(w)\,dw$ ← wlog follow sides of rectangle $z \to z_0$

- Now smuggle $f(z)$ into the integrand –

$$\int_z^{z+\Delta z} f(w)\,dw = \underbrace{\int_z^{z+\Delta z} f(w)\,dw - \int_z^{z+\Delta z} f(z)\,dw}_{\int_z^{z+\Delta z} f(w)-f(z)\,dw} + \underbrace{f(z)\int_z^{z+\Delta z} dw}_{f(z)\cdot\Delta z}$$

same

$$= \int_z^{z+\Delta z} f(w)-f(z)\,dw + f(z)\Delta z.$$

But

$$\left|\int_z^{z+\Delta z} f(w)-f(z)\,dw\right| \le \underbrace{\sqrt{2}\,|\Delta z|}_{|C_{\Delta z}|}\ \underbrace{\operatorname*{Max}_{w\in R_{\Delta z}} |f(w)-f(z)|}_{\text{max mod of integrand}} = o(1)\Delta z$$

cont $f \Rightarrow o(1) \xrightarrow[\Delta z\to 0]{} 0$

So

$$\int_z^{z+\Delta z} f(w)\,dw = o(1)\Delta z + f(z)\Delta z$$

Conclude:

$$F'(z) = \lim_{\Delta z\to 0} \frac{1}{\cancel{\Delta z}}\left[\int_z^{z+\Delta z} f(z)\,dz = o(1)\cancel{\Delta z} + f(z)\cancel{\Delta z}\right]$$

tends to zero

$= f(z)$ as Claimed. ✓

This Completes (II) and hence proves (C–G)!

The main consequence of the (C-G) Theorem is the **Cauchy Integral Formula**: If $f'(z)$ exists, then

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{w-z}\, dw. \qquad \text{(CIF)}$$

$\oint_{\mathcal{C}} \frac{dz}{z} = 2\pi i$

$\mathcal{C}$: positively oriented closed curve which winds around z exacty once.

- If $g(w,z) = \frac{f(w)}{w-z}$ and $\frac{\partial g}{\partial z}$ are continuous in an open set containing $\mathcal{C}$, a theorem in advanced calculus tells us we can diff wrt z thru $\int$-sign:

HW

$$f'(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\partial}{\partial z} \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{(w-z)^2}\, dw$$

$$\vdots$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{(w-z)^{n+1}}\, dw$$

Conclude: $f: D \to \mathbb{C}$ analytic in $D \Rightarrow f^{(n)}(z)$ exists and cont in D $\forall\, n \in \mathbb{N}$!

- Applications of (CIF): T-series, Laurent Series, Residue Thm, Fundamenta Thm Algebra, Liouville's Thm, Max Principle, Elliptic Regularity for soln's $\Delta u = 0$, Morera's Thm, ...

Next Topic: Applications of (CIF):