VII Applications of the Cauchy Integral Formula

- Recall the (CIF) provides an integral formula for an analytic function $f(z)$ in terms of integrated values of f along any curve $\mathcal{C}$ which winds around z. (Assumes only $f'(z)$ exists in nbhd of $\mathcal{C}$)

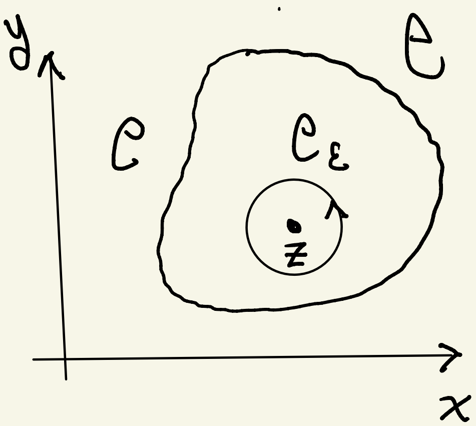
$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{w-z}\, dw \qquad \text{(CIF)}$$

Circle indicates $\mathcal{C}$ is a pos. oriented simple closed curve

comes from $\oint_{\mathcal{C}} \frac{dw}{w-z}\, dw = 2\pi i$

Integral the same for every positively oriented $\mathcal{C}$ which winds around z once.

- I.e.,



$$\oint_{\mathcal{C}} \frac{f(w)}{w-z}\, dw = \int_{\mathcal{C}_\varepsilon} \frac{f(w)}{w-z}\, dw$$

$$= f(z) \int_{\mathcal{C}_\varepsilon} \frac{1}{w-z}\, dw + o(1)$$

$w(t) = z + \varepsilon e^{it}, \quad 0 \leq t \leq 2\pi$

$dw = i\varepsilon e^{it}$

$$= f(z) \int_0^{2\pi} \frac{i\cancel{\varepsilon}\cancel{e^{it}}}{\cancel{\varepsilon}\cancel{e^{it}}}\, dt + o(1)$$

$$= f(z) \cdot 2\pi i \quad \checkmark$$

②

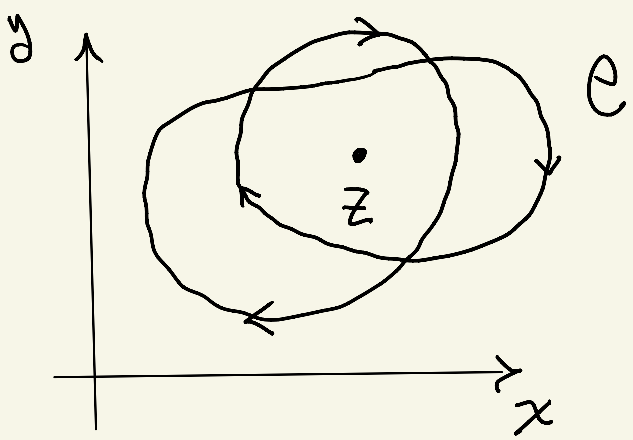
- **Recall**: For general closed curves $\mathcal{C}$ which can cross themselves,

$$I(\mathcal{C};z_0) \equiv \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{dw}{w-z_0} = \text{"winding \# of } \mathcal{C} \text{ around } z_0\text{"}$$

So more generally →

HW $$f(z_0) = \frac{I(\mathcal{C};z_0)}{2\pi i}\int_{\mathcal{C}} \frac{f(w)}{w-z_0}\,dw$$ (General CIF)

Picture:



$I(\mathcal{C};z) = -2$

- From here on, we assume $\mathcal{C}$ is a simple closed curve (s.c.c), a closed curve which does not cross itsel $\Rightarrow I(\mathcal{C};z) = \pm 1, 0 \Rightarrow$ (CIF)

f analytic $\Rightarrow$ $$\boxed{f(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(w)}{w-z}\,dw}$$

- Since $g(w,z) = \frac{f(w)}{w-z}$ is continuous along $\mathcal{C}$ and continuously differentiable in z, Math 127A (real analysis "Lebesgue Dominated Convergence") implies you can differentiate thru $\int$-sign

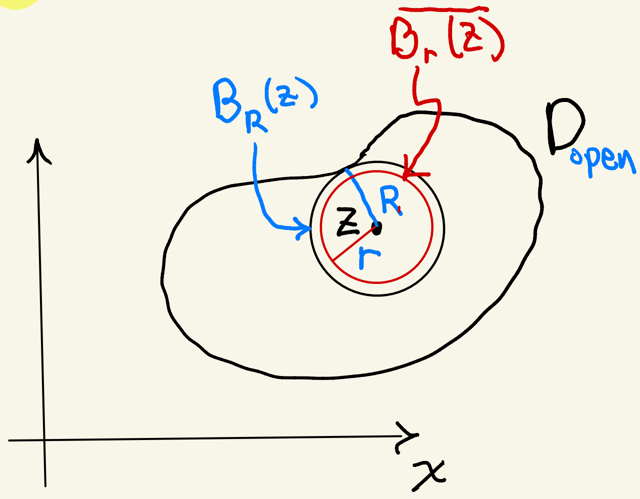
HW
$$f'(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{(w-z)^2}\, dw$$

$$\vdots$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{(w-z)^{n+1}}\, dw$$

- Assume now that $f: D \to \mathbb{C}$ is analytic in an open set $D \subseteq \mathbb{C}$, so that for every $z \in D$ there exists $r > 0$ st $B_r(z) \subseteq D$. Let R be the largest $r > 0$ st $B_r(z) \subseteq D$. Recall $\overline{B_r(z)}$ = closure of $B_r(z)$ $= B_r(z) \cup \partial B_r(z)$. Now $\overline{B_R(z)}$ is not in D_{open}, but for all $r' < r$, we have

$$\overline{B_{r'}(z)} = \bigcap_{r'' > r'} B_{r''}(z) \subseteq D_{open}.$$



Conclude: If $\overline{B_r(z)} \subseteq D$, then f analytic is continuous on the compact set $\overline{B_r(z)}$, and hence $|f(z)|$ takes a maximum value M in $\overline{B_r(z)}$. (a continuous real valued function takes its max and min values on a compact set)

We have:

Theorem (Weak Max Principle) If f is analytic in a nbhd of $\overline{B_r(z)}$, then $\exists M>0$ such that

$$|f(z)| \le M \quad z \in \overline{B_r(z)}.$$

- We can now state Taylor's Theorem, but paspone the proof. (The proof is the <u>same</u> as in Math 21C)

Theorem (Taylors Thm). The Taylor Series

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

converges in the largest open ball $B_R(z_0)$ in which f is analytic, converges uniformly in $\overline{B_r(z_0)}$ for all $r<R$, and diverges for $|z-z_0|>R$. R = Radius of Convergence.

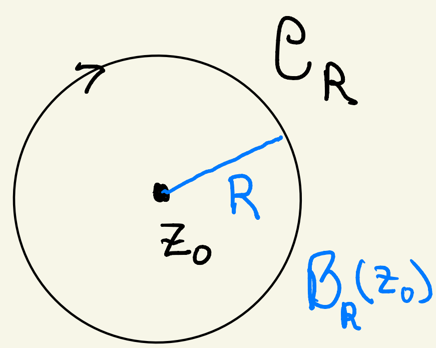
- To prove Taylors Thm, we need the following estimate on $|f^{(n)}(z)|$ (which follows from (CIF)):

Theorem: (Cauchy's Inequality) Assume f is analytic in a nbhd of $\overline{B_R(z_0)}$. Then

$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n} M, \quad M = \max_{z \in \overline{B_R(z_0)}} |f(z)| \qquad \text{(C Ineq)}$$

Proof: Let C_R denote the circle of radius R, center z_0, so $C_R = \partial B_R(z_0) \equiv$ boundary of $B_R(z_0)$.

Then (CIF) $\Rightarrow$



$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(w)}{(w-z_0)^{n+1}}\, dw.$$

Thus:

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \left| \int_{C_R} \frac{f(w)}{w-z_0}\, dw \right| \leq \frac{n!}{2\pi} \int_{C_R} \frac{|f(w)|}{|w-z_0|^{n+1}} |dw|$$

$ds = \sqrt{dx^2 + dy^2}$

$$\leq \frac{n!}{2\pi} \int_{C_R} \frac{M}{R^{n+1}} |dw| \leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} |C_R|$$

$$\leq \frac{n!}{\cancel{2\pi}} \frac{M}{R^{n+\cancel{1}}} \cancel{2\pi} \cancel{R} \leq \frac{n!}{R^n} M \quad \checkmark$$

• **Cor ①** (Liouville's Theorem – Big) Every bounded entire function is constant.

(f entire if analytic, $f'(z)$ exists, $\forall z \in \mathbb{C}$)

Proof: Assume f entire and $|f(z)| \le M$ $\forall z \in \mathbb{C}$. We prove $f(z) = \text{const}$. (Thus there are no non-constant analytic functions which are entire & bounded)

But f analytic in $B_R(z) \Rightarrow$ (by (C-Ineq) case $n=1$),

$$|f'(z)| \le \frac{1}{R} M, \quad |z| < R.$$

Since f is entire, this holds $\forall R$. But the only way we can have $|f'(z)| \le \frac{M}{R}$ $\forall R > 0$ is if $|f'(z)| = 0$, so $f'(z) = 0 \Rightarrow f(z) = \text{const}$. ✓

Conclude: Every analytic function f which has a derivative at every $z \in \mathbb{C}$, (like e^z, $\sin z$, $\cos z$, z^n etc), must have the property that $|f(z)| \to \infty$ as $|z| \to \infty$, at least along some subsequence $z = z_k$.

Thus $\cos z$ could not be extended to a bounded analytic fn!

• **Cor (2) (Fundamental theorem of Algebra – Big)**

Let $P(z)$ be any complex polynomial of degree $n \geq 1$,
$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, $a_k \in \mathbb{C}$, $a_n \neq 0$. (P_n)
Then $P(z)$ has at least one root $z_0 \in \mathbb{C}$ where

$$P(z_0) = 0.$$

Note: By the Remainder Thm for polynomials, this implies $P(z) = (z - z_0)\, q(z)$ where $\deg(q) < \deg(P)$. I.e., the Remainder Thm for complex polyn's holds for the same algebraic reasons it holds for $\mathbb{N} \subseteq \mathbb{R}$; i.e., given polyn's P_n, P_m, $1 \leq m < n$ in $\mathbb{N}$, there always exist polynomials $q(z)$ and $r(z)$ st $\underbrace{P_n(z)}_{\deg n} = \underbrace{P_m(z)}_{\deg m}\, q(z) + r(z)$, where

$\deg r < \deg P_m = m$. Taking $P_m(z) = (z - z_0)$, we get $P(z) = (z - z_0)\, q(z) + \text{Const}$, & Const $= 0$ as $P(z_0) = 0$.

• Since $P(z_0) = 0 \Rightarrow P(z) = (z - z_0) P_{n-1}(z)$, applying this to $P_{n-1}(z), P_{n-2}(z), \ldots, P_1(z)$ we conclude that every complex polyn P_n is product of n linear factors

Proof (Fundamental Theorem Algebra):

- Assume for contradiction that there does not exist z_0 st $P(z_0)=0$, where $P(z)$ is given in (P_n), $n \geq 1$. We prove that this assumption leads to the conclusion $P(z) = \text{const}$, not true, so the assumption cannot be true, implying $P(z_0)$ must be zero for some $z_0 \in \mathbb{C}$.

Now if $P(z) \neq 0$, then $f(z) = \frac{1}{P(z)}$ is a (non-constant) entire function. We obtain a contradiction by showing that if $\frac{1}{P(z)}$ is entire, then it is bounded, and by Louiville Thm, constant ※.

To prove $\frac{1}{P(z)}$ bded if it is entire, note that

$$\lim_{z \to \infty} |P(z)| = \infty.$$

That is, for $z = re^{i\theta}$, $r = |z|$, the $a_n z^n$ term "wins" as $r \to \infty$. (HW make a careful proof - see below.) Thus there exists $R > 0$ st if $|z| > R$, then $|P(z)| > 1$, $\frac{1}{|P(z)|} < 1$.

Thus for $|z| > R$ we have

$$\left|\frac{1}{p(z)}\right| = \frac{1}{|p(z)|} \leq 1, \quad |z| > R.$$

But "$|z| \leq R$" is the set $\overline{B_R(0)}$, a compact set, so if $\frac{1}{p(z)}$ is analytic, then it is continuous, and hence bounded on $|z| \leq R$, so

$$\left|\frac{1}{p(z)}\right| = \frac{1}{|p(z)|} \leq M, \quad |z| \leq R.$$

Thus $\left|\frac{1}{p(z)}\right| \leq \begin{cases} 1 & |z| > R \\ M & |z| \leq R \end{cases}$, so $f(z) = \frac{1}{p(z)}$ is bounded. By Liouville Thm, a bounded analytic function is constant, $\times$ contradiction $n \geq 1$. Conclude our assumption $p(z) \neq 0$ must be false, and we have proven $p(z_0) = 0$ for some $z_0 \in \mathbb{C}$ by <u>contraction</u>. ✓

<u>Cor</u>: Every <u>real</u> polynomial can be factored into linear and irreducible quadratic factors.

Proof of Cor: $p(z)$ in (P_n) is real if the coefficients a_n are real, $a_n \in \mathbb{R}$. (Ie, then $P_n(x)$ restricts to a real polynomial.) In this case, if $P_n(z) = 0$ then $P_n(\bar{z}) = 0$. This follows by properties of the complex conjugate, i.e.

$$\bar{0} = \overline{P_n(z)} = \overline{a_0 + a_1 z + \cdots + a_n z^n} = \bar{a}_0 + \bar{a}_1 \bar{z} + \cdots + \bar{a}_n \bar{z}^n$$
$$= a_0 + a_1 \bar{z} + \cdots + a_n \bar{z}^n = P(\bar{z}) \text{ since } a_n = \bar{a}_n.$$

Thus all complex factors in $p(x)$ pair as conjugates,

$$(x - z_n)(x - \bar{z}_n) = x^2 - (z_n + \bar{z}_n) + z_n \bar{z}_n = x^2 - 2x_n + (x_n^2 + y_n^2)$$

where $z_n = x_n + i y_n$. Conclude: $P_n(z) = \prod_{n=1}^{n} a_n (x - z_n)$, $(x - z_n)$ real and linear if $z_n \in \mathbb{R}$, or pairwise form irreducible quadratics when $z_n \in \mathbb{C}$.

HW Give a careful proof of:

Lemma: For $P(z) = P_n(z)$ given in (P_n), $n \geq 1$,

$\exists R > 0$ st $|z| > R \Rightarrow |P(z)| \geq 1$.

Proof (Hint) Let $z = re^{i\theta}$, $|z| = r > 1$. Then

$$|P(z)| = |a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n|$$
$$\geq |a_n z^n| - |a_0| - |a_1 z| - \cdots - |a_{n-1} z^{n-1}|$$
$$\geq (\underbrace{|a_n| r - n a}_{\geq 1 \text{ for } r > \frac{na+1}{|a_n|}}) r^{n-1}, \quad a = |a_0| + \cdots + |a_{n-1}|$$

(Uses $a_n \neq 0$)

- We are considering whether or not there are "low regularity" analytic functions, meaning f' exists but not continuous, and we've shown the answer is NO! whenever $f'(z)$ exists, $f: D_{open} \to \mathbb{C}$, f has derivatives of all orders in a nbhd of every point, and if D simply connected, f has anti-derivatives of all orders, and FTC holds. In this case, the integral of f around every closed curve is zero.
- We now ask whether it goes the other way; namely, if $f: D_{open} \to \mathbb{C}$ has the property that the integral around every closed path is zero, is f analytic? I.e., must $f'(z)$ exist $\forall z \in D$? Morera's Theorem states Yes, so long as f is at least continuous –

<u>Theorem</u> (<u>Morera</u>): If $f: D \to \mathbb{C}$ is continuous in an open set $D \subseteq \mathbb{C}$, and $\oint_{\mathcal{C}} f(z)\,dz = 0$ for every closed curve in D, then f is analytic in D. – No need to assume D simply connected –

(Essentially $\frac{1}{z}$ is ruled out by $\oint_{\mathcal{C}} f(z)\,dz = 0$!)

Proof: Assume $\oint_{\mathcal{C}} f(z)\,dz = 0$ for all $\mathcal{C} \subseteq D$.

Then $\int_{\mathcal{C}} f(z)\,dz$ is independent of path $\mathcal{C}$ taking $A \to B$, and we can write $\int_{\mathcal{C}} f(z)\,dz = \int_A^B f(z)\,dz$. Thus define

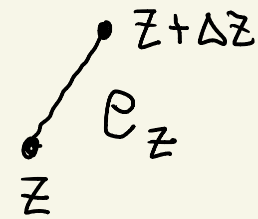
$$F(z) = \int_A^z f(w)\,dw,$$

well defined for any $\mathcal{C} \subseteq D$, $\mathcal{C}: A \to B$.

Claim: $F'(z) = f(z)$. For this we estimate

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| = \left| \frac{1}{\Delta z} \underbrace{\left\{ \int_A^{z+\Delta z} f(w)\,dw - \int_A^{z} f(w)\,dw \right\}}_{\text{wlog assume same path } A \to z} - f(z) \right|$$

$$= \left| \frac{1}{\Delta z} \int_z^{z+\Delta z} f(w)\,dw - f(z) \right| \qquad \frac{1}{\Delta z} \int_z^{z+\Delta z} \underbrace{f(z)}_{\text{constant}}\,dw$$



$$= \left| \frac{1}{\Delta z} \int_z^{z+\Delta z} f(w) - f(z)\,dw \right|$$

$$\leq \frac{1}{|\Delta z|}\, |\mathcal{C}_z| \underbrace{\max_{z \in \mathcal{C}_z} |f(w) - f(z)|}_{o(1) \text{ as } \Delta z \to 0} \qquad |\mathcal{C}_z| = |\Delta z|$$

$$\leq \frac{1}{\cancel{|\Delta z|}} \cancel{|\Delta z|}\, o(1) \xrightarrow[\Delta z \to 0]{} 0 \quad \Rightarrow \quad F'(z) = f(z)$$

and (CIF) $\Rightarrow$ $F''(z) = f'(z)$ continuous $\Rightarrow$ f analytic ✓

- An important Corollary of Morera's Thm tells us when apparant singularities in a function are removable:

Cor: If f is continuous in an open set $D \underset{\text{open}}{\subseteq} \mathbb{C}$, and $f'(z)$ exists at every $z \in D$ except at one point $z_0 \in D$, then $f'(z_0)$ exists as well

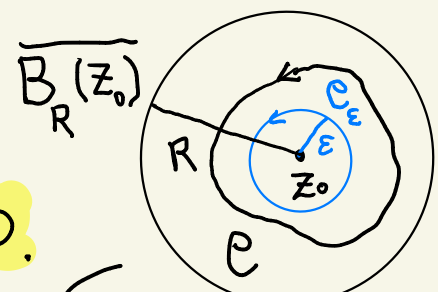
Proof: By Morera's Theorem it suffices to prove that $\oint_{\mathcal{C}} f(z)\,dz = 0$ for every closed curve in some neighborhood of z_0. To this end, note that since D is open, $\exists\, R>0$ st $\overline{B_R(z_0)} \subseteq D$. Since $f: \overline{B_R(z_0)} \to \mathbb{C}$ continuous and $\overline{B_R(z_0)}$ compact, it follows that f is bounded on $\overline{B_R(z_0)}$

$\Rightarrow$ $\exists\, M>0$ such that $|f(z)| \leq M \quad \forall z \in \overline{B_R(z_0)}$.
implies

So $\left|\oint_{\mathcal{C}} f(z)\,dz\right| = \left|\oint_{\mathcal{C}_\varepsilon} f(z)\,dz\right|$

$\mathcal{C}_\varepsilon = \partial B_\varepsilon(z_0)$
$\varepsilon < R$

$\leq |\mathcal{C}_\varepsilon| M \xrightarrow[\varepsilon \to 0]{} 0.$ ✓



HW Use Morera's Theorem to prove that the uniform limit of analytic functions is analytic.

Hint: You can always pass an integral sign thru a sequence of uniformly converging functions. I.e., $f_n \to f$ uniformly in $D \Rightarrow$

$$\lim_{n\to\infty} \int_{\mathcal{C}} f_n(z)\,dz = \int_{\mathcal{C}} \lim_{n\to\infty} f_n(z)\,dz = \int_{\mathcal{C}} f(z)\,dz$$

• **Maximum Principle**: "An analytic function always takes its max modulus on the boundry".

Theorem (Max Principle) Assume f is analytic in a nbhd of a simple closed curve $\mathcal{C}$, and let D denote the open set inside $\mathcal{C}$, so $\mathcal{C} = \partial D$, $\bar{D} = D \cup \mathcal{C}$. Then the maximum modulus $|f(z)|$ of f in $\bar{D}$ is taken on the boundary $\partial D = \mathcal{C}$. I.e.,

$$M_\partial \equiv \max_{z \in \partial D} \{|f(z)|\} \geq |f(w)| \quad \text{for all } w \in \bar{D}.$$

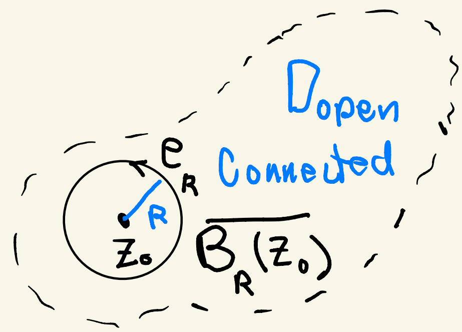
(In fact, if $f(z) \neq 0$ in D, then f takes its minimum modulus on boundary of D as well. We do max case)

For the proof we assume for now the following

Lemma: If f is non-constant in D (i.e., $f(z_1) \neq f(z_2)$ for some $z_1 \neq z_2$ in D) then $|f(z)|$ is non-constant in every nbhd of a point $z \in D$.

This follows directly from Taylor's Thm; ie, f constant in a nbhd $\Rightarrow$ T-series around every pt in nbhd is constant $\Rightarrow$ extends to D because D connected - later

Proof of Maximum Modulus Thm: So wlog assume D connected and f non-constant in D, so Lemma applies. Assume for contradiction that the maximum value of $|f(z)|$ in $\overline{D}$ occurs at an interior point $z_0 \in D$. Then D open implies that for R sufficiently small, $\overline{B_R(z_0)} \subseteq D$. We now apply the (CIF) to $\overline{B_R(z_0)}$.



$$\text{(CIF)} \Rightarrow f(z_0) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(w)}{w - z_0}\, dw, \quad C_R = \partial \overline{B_R(z_0)}.$$

Letting $z(t) = z_0 + Re^{it}$, $0 \le t \le 2\pi$ $\Rightarrow$

$$f(z_0) = \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(z_0 + Re^{it})\, \cancel{i} \cancel{R} \cancel{e^{it}}}{\cancel{R} \cancel{e^{it}}}\, dt$$

So

$$\boxed{f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt}$$

"$f(z_0)$ is the complex average of its values on $\partial D = C$"

Thus

$$|f(z_0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + Re^{it})|\, dt.$$

Thus we have:

$$|f(z_0)| \le \underbrace{\frac{1}{2\pi}\int_0^{2\pi} |f(z_0+Re^{it})|\,dt}_{\text{The average of } |f(z_0)| \text{ on } \partial B_R(z_0)}$$

But the average of N numbers is always $\le$ maximum of the N numbers — and since the average of a function is the limit of an average given by a Riemann sum, the average of a real valued function over $[a,b]$ is always $\le$ its maximum on $[a,b]$. Thus

$$|f(z_0)| \le \underset{z\in\partial B_R}{\text{Max}} |f(z)|.$$

But our assumption is that $|f(z_0)|$ is max, so

$$|f(z_0)| \ge \underset{z\in\partial B_R}{\text{Max}} |f(z)|.$$

Thus

$$|f(z_0)| = \underset{z\in\partial B_R}{\text{Max}} |f(z)|$$

That is, our assumption is that

$$|f(z_0)| \geq |f(z_0+Re^{it})| \quad \forall\, t \in [0, 2\pi)$$

so

$$|f(z_0)| \geq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+Re^{it})|\,dt$$

but we proved

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+Re^{it})|\,dt.$$

Thus

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0+Re^{it})|\,dt = |f(z_0)| = \max_{0 \leq t \leq 2\pi} |f(z(t))|$$

The only way the average of a function can equal its maximum is if the function is constant. Thus

$$|f(z_0)| = |f(z_0+Re^{it})| \quad \forall\, t \in [0, 2\pi).$$

But by same argument, this holds $\forall\, r \leq R$, and hence $|f(z_0)| = |f(z)|$ $\forall\, z \in B_R(z_0)$. So by the Lemma, $f(z) = \text{const}$ in D (because a non-constant function can't have $|f(z)| = \text{const}$ in a nbhd of any pt.). This ※ our assumption, & Theorem is proved ✓

HW Average of real valued function –

Recall: The average of N real numbers $x_1, \dots, x_N$ is

$$\bar{x} = \frac{x_1 + \cdots + x_N}{N} = \frac{1}{N}\sum_{k=1}^{N} x_k .$$

Defn: The average of $f : [a,b] \to \mathbb{R}$ is

$$\bar{f} = \frac{1}{|b-a|}\int_a^b f(x)\,dx .$$

Show that $\bar{f}$ really is an average in a limiting sense –

Soln: $\int_a^b f(x)\,dx \approx \sum_{k=1}^{N} f(x_k)\Delta x \qquad \Delta x = \frac{b-a}{N}$

$$\frac{1}{|b-a|}\int_a^b f(x)\,dx \cong \frac{1}{N\cancel{\Delta x}} \sum_{k=1}^{N} f(x_k)\cancel{\Delta x}$$

$$\cong \frac{1}{N}\sum_{k=1}^{N} f(x_k) \equiv \text{average}$$

HW Prove that if $f(x) \geq 0$, then

$$0 \leq \bar{f} \leq \frac{1}{|b-a|}\int_a^b f(x)\,dx .$$

Harmonic Functions:

Recall: $f(z) = u(x,y) + iv(x,y)$ is analytic in D iff [C-R] holds in D:

$$u_x = v_y, \qquad u_y = -v_x.$$

Differentiating gives:

$$u_{xx} = v_{yx} \qquad v_{xx} = -u_{yx}$$

$$u_{yy} = -v_{xy} \qquad v_{yy} = u_{xy}$$

$$u_{xx} + u_{yy} = 0 \qquad v_{xx} + v_{yy} = 0$$

Defn: The Laplacian of a real valued function $u: \mathbb{R}^2 \to \mathbb{R}$ is defined to be

$$\underbrace{\Delta}_{\text{Laplacian}} u = u_{xx} + u_{yy}.$$

Defn: A real valued function $u(x,y)$ is said to be "harmonic" if

$$\Delta u = 0.$$

We have thus proven:

Theorem: The real and imaginary parts of an analytic function are harmonic.

Defn: If $\Delta u = 0$ and $\Delta v = 0$ and $f(z) = u + iv$ is analytic, then we say u, v are harmonic conjugates.

- The study of solutions of $\Delta u = 0$ is part of the subject of Partial Differential Equations (PDE). In the case of $\mathbb{R}^2$, we can use complex variables to prove theorems about solutions of $\Delta u = 0$, i.e., theorems in PDE.

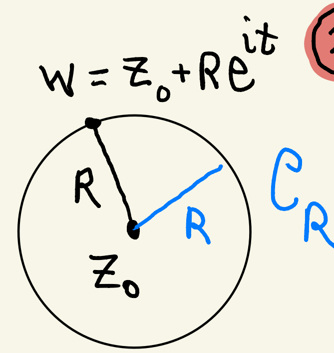
Theorem: An harmonic function $u(x,y)$, i.e., a solution of $\Delta u = 0$, is equal to its average on any circle of radius R:

$$u(\underset{\sim}{x}_0) = \frac{1}{2\pi} \int_0^{2\pi} u(\underset{\sim}{x}_0 + Re^{it})\, dt$$

slight abuse of notation

Proof: By (CIF) we have

$$f(z_0) = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w - z_0}\,dw$$



Evaluating by parameterization gives

$$w(t) = z_0 + Re^{it} \qquad 0 \le t \le 2\pi$$

$$dw = iRe^{it}dt$$

$$f(z_0) = \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{f(z_0+Re^{it})\cancel{i}\cancel{R}\cancel{e^{it}}}{\cancel{R}\cancel{e^{it}}}\,dt$$

$$\boxed{f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+Re^{it})\,dt}$$

"$f(z_0)$ is the complex average of its values on $\partial D = C$"

Thus

$$f(z_0) = u(\underset{\sim}{x}_0) + iv(\underset{\sim}{x}_0) = \frac{1}{2\pi}\int_0^{2\pi} u(\underset{\sim}{x}_0+Re^{it}) + iv(\underset{\sim}{x}_0+Re^{it})\,dt$$

and equating real and imaginary parts gives

$$u(\underset{\sim}{x}_0) = \frac{1}{2\pi}\int_0^{2\pi} u(\underset{\sim}{x}_0+Re^{it})\,dt$$

$$v(\underset{\sim}{x}_0) = \frac{1}{2\pi}\int_0^{2\pi} v(\underset{\sim}{x}_0+Re^{it})\,dt.$$

Conclude: The theorem is proven if we can show that for every harmonic $u(x,y)$ there exists an harmonic conjugate $v(x,y)$, i.e. such that $u+iv$ is analytic.

Theorem (complex Conjugate) For every $u(x,t)$ in C^2 satisfying $\Delta u = 0$ in D_{open} and s.c., there exists an harmonic conjugate $v(x,t)$ such that $\Delta v = 0$ and $f(z) = u + iv$ is analytic in D.

Since the real and imaginary parts of an analytic function are C^∞ and locally expandable into Taylor Series, we have an immediate corollary, which is a PDE regularity result for solns of the $\Delta u = 0$.

Cor: Every C^2 solution of $\Delta u = 0$ is C^∞. Furthermore, u is locally expandable into T-series, with radius of convergence R = distance from the center to the nearest singularity in $f(z) = u + iv$.

Proof of Theorem Complex Conjugate: Assume $u \in C^2$ and $\Delta u = 0$. We find $v(x,t)$ such that $\Delta v = 0$ and $f(z) = u + iv$ is analytic. By (C-R) it suffices to find $v(x,t)$ s.t. $\Delta v = 0$ and $u_x = v_y$, $u_y = -v_x$. To this end define

$$g(x,y) = \underbrace{u_x}_{U} - i\underbrace{u_y}_{V} = U + iV.$$

I.e., we know what v_x & v_y must be, so taking $\Delta z = \Delta x$ we know what $f'(z)$ must be!

Claim: g is analytic. For this it suffices to show (C-R) $U_x = V_y$, $U_y = -V_x$. But

$$\left.\begin{array}{l} U_x = u_{xx},\ U_y = u_{xy} \\ V_x = -u_{yx},\ V_y = -u_{yy} \end{array}\right\} \text{ and } \Delta u = 0 \Rightarrow u_{xx} = -u_{yy}.$$

Thus $U_x = u_{xx} = -u_{yy} = V_y$, $U_y = u_{xy} = u_{yx} = -V_x$ ✓C-R

Since g analytic in a s.c. domain, it has an antiderivative $F(z) = \hat{u} + i\hat{v}$ analytic.

But $F'(z) = \hat{u}_x + i\hat{v}_x = U + iV = u_x - iu_y$ analytic, and $\hat{u}_x = u_x$, $\hat{u}_y = -\hat{v}_x = -V = u_y \Rightarrow \nabla \hat{u} = \nabla u$

$\Rightarrow \hat{u} = u + \text{Const} \Rightarrow f(z) = u + i\hat{v} = F(z) + \text{Const}$ is analytic

$\Rightarrow v(x,t) = \hat{v}(x,t)$ is an harmonic conjugate of u. ✓

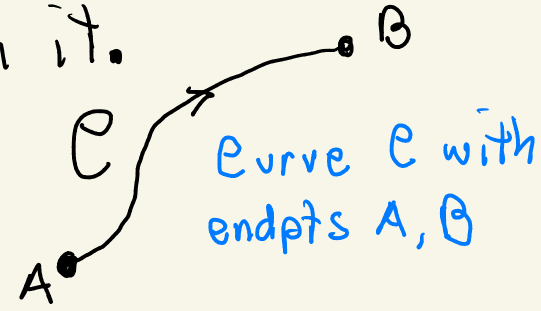
HW Find $u(x,y)$ and $v(x,y)$ st $\Delta u = 0$, $\Delta v = 0$ but $f(z) = u + iv$ is NOT analytic.

Final Comments on PDE's:

- The 3 1st order operators of classical physics are: Gradient (∇), Curl and Divergence (Div). These were the topic of Math 21D. Each one has an extension of FTC based on it.

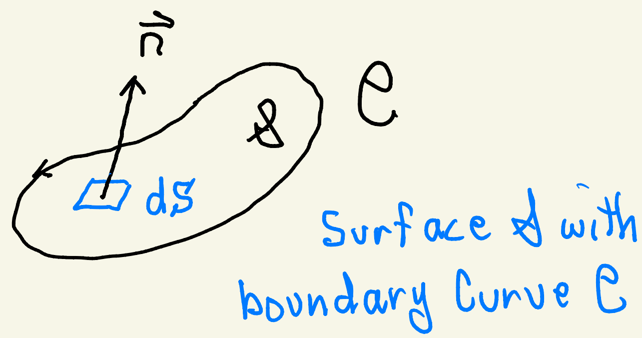
(1) (Conservation of Energy)

$$\int_{\mathcal{C}} \nabla f \cdot \vec{T}\, ds = f(B) - f(A)$$



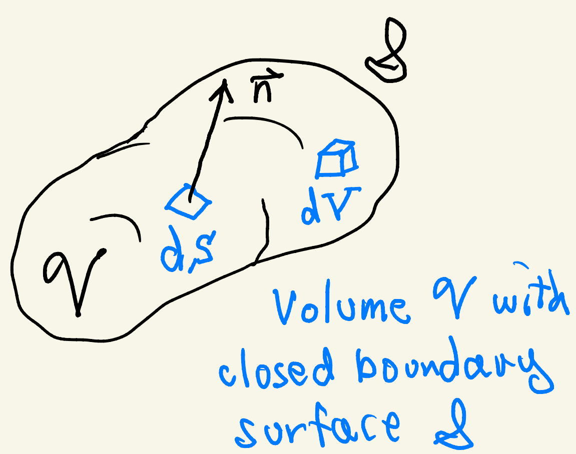
(2) Stokes Theorem:

$$\int_{\mathcal{S}} \text{Curl}\, \vec{F} \cdot \vec{n}\, dS = \oint_{\mathcal{C}} \vec{F} \cdot \vec{T}\, ds \qquad \mathcal{C} = \partial \mathcal{S}$$



(3) Divergence Theorem:

$$\iiint_{\mathcal{V}} \text{Div}\, \vec{F}\, dv = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS \qquad \mathcal{S} = \partial \mathcal{V}$$



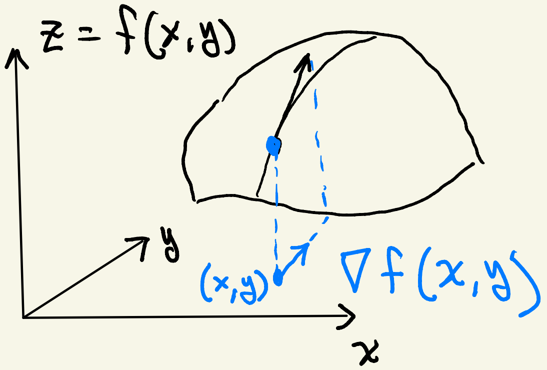
Complex Variables Gives Us a 4'th (Complex Version of (1))

(4) $\int_{\mathcal{C}} f'(z)\, dz = f(B) - f(A)$

Formulas and Meanings:

- $\nabla f = \overrightarrow{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)}$

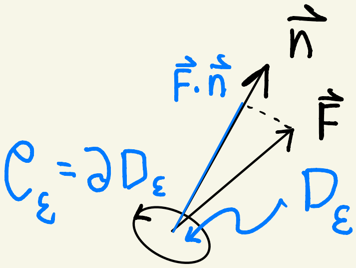
Points in direction of steepest increase of f, length $= \frac{df}{ds}$ along ∇f



$\nabla f \cdot \vec{v} = \frac{df}{ds}$ in direction $\vec{v}$

- $\text{Curl}\,\vec{F} \equiv \underset{\text{Del cross }\vec{F}}{\vec{\nabla} \times \vec{F}} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = (P_y - N_z)\underset{\sim}{i} - (P_x - M_z)\underset{\sim}{j} + (N_x - M_y)\underset{\sim}{k}$

Points in direction of max'al circulation per area, and $\text{Curl}\,\vec{F} \cdot \vec{n}$ = circulation/area around axis $\vec{n}$



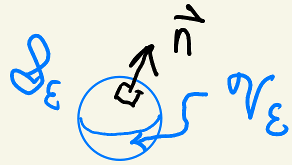
$\vec{F} = \overrightarrow{(M, N, P)}$

$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$$\text{Curl}\,\vec{F} \cdot \vec{n} = \lim_{\varepsilon \to 0} \frac{\oint_{C_\varepsilon} \vec{F} \cdot \vec{T}\, ds}{\text{Area}(D_\varepsilon)} = \frac{\text{circulation}}{\text{area}}$$

- $\text{Div}\,\vec{F} = \underset{\text{Del dot }\vec{F}}{\vec{\nabla} \cdot \vec{F}} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$

Gives the (outward) Flux per volume at a point



$$\text{Div}\,\vec{F} = \lim_{\varepsilon \to 0} \frac{\iint_{S_\varepsilon} \vec{F} \cdot \vec{n}\, dS}{\text{Vol}(V_\varepsilon)} = \frac{\text{Flux}}{\text{Vol}}$$

* Ex: $\vec{F} = \underset{\sim}{\rho}\, \underset{\sim}{\vec{v}}$ = mass Flux Vector

density $= \frac{\text{mass}}{\text{vol}}$ velocity

$\Rightarrow \iint_S \vec{F} \cdot \vec{n}\, dS$ = "Flux" = mass/time out thru S

- Relationship between Them:

$$\begin{array}{ccccccc} \mathbb{R} & \xrightarrow{\vec{\nabla}} & \mathbb{R}^3 & \xrightarrow{\text{Curl}} & \mathbb{R}^3 & \xrightarrow{\text{Div}} & \mathbb{R} \\ \uparrow f & & \uparrow \vec{F} & & \uparrow \vec{F} & & \uparrow f \\ \mathbb{R}^3 & & \mathbb{R}^3 & & \mathbb{R}^3 & & \mathbb{R}^3 \end{array}$$

Two in a row make zero: $\text{Curl}\,\nabla f = 0$, $\text{Div}\,\text{Curl}\,\vec{F} = 0$

(both by equality of mixed partials: $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, etc)

HW ① Verify $\text{Curl}\,\nabla f = 0$, $\text{Div}\,\text{Curl}\,\vec{F} = 0$

② Show: $\text{Div}\,\nabla f = \Delta f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$

- The Laplacian is the most important 2nd order operator of classical physics. Virtually all equations of classical physics are formulated in terms of ∇, Curl, Div, Δ.

- Three 2nd order Linear equations of classical physics:

Laplaces Eqn: $\Delta u = 0$ (Elliptic)

Heat Equation: $u_t - c^2 \Delta u = 0$ (Parabolic)

Wave Equation: $u_{tt} - c^2 \Delta u = 0$ (Hyperbolic)

Solutions of $\Delta u = 0$ represent "steady state" soln's.
Most processes settle down to steady state as $t \to \infty$.