75. Integration Around a Branch Point

For our final illustration of the use of the residue theorem in evaluating real integrals, we now consider an example involving branch points and branch cuts.

Let x^{-a}, where $x > 0$ and $0 < a < 1$, denote the principal value of the indicated power of x; that is, x^{-a} is the positive real number $\exp(-a \operatorname{Log} x)$. We shall evaluate the improper real integral

$$\int_0^\infty \frac{x^{-a}}{x+1}\,dx \qquad (0 < a < 1) \tag{1}$$

which is important in the study of the gamma function.[1] The integral exists when $0 < a < 1$ because the integrand behaves like x^{-a} near $x = 0$ and like x^{-a-1} as x tends to infinity.

To evaluate integral (1), we consider the two line integrals

$$\int_{C_1} f_1(z)\,dz, \qquad \int_{C_2} f_2(z)\,dz$$

where

$$f_1(z) = \frac{z^{-a}}{z+1} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right),$$

$$f_2(z) = \frac{z^{-a}}{z+1} \qquad \left(|z| > 0, \frac{\pi}{2} < \arg z < \frac{5\pi}{2}\right)$$

and C_1 and C_2 are the simple closed contours shown in Fig. 52. In that figure $\rho < 1 < R$, and the angle ϕ is chosen so that $\pi/2 < \phi < \pi$.

Observe that the function f_1 is analytic within and on C_1; hence

$$\int_{C_1} f_1(z)\,dz = 0. \tag{2}$$

Moreover, the function f_2 is analytic within and on C_2 except for the simple pole at the point $z = -1$ which is interior to C_2. Now in the definition of f_2,

$$z^{-a} = \exp\left[-a(\text{Log}\,|z| + i \arg z)\right] \qquad \text{where} \qquad \frac{\pi}{2} < \arg z < \frac{5\pi}{2},$$

and the residue of f_2 at $z = -1$ is

$$\lim_{z \to -1} (z+1) f_2(z) = \lim_{z \to -1} z^{-a} = \exp(-a\pi i).$$

[1] See, for example, p. 4 of the book by Lebedev cited in Appendix 1.

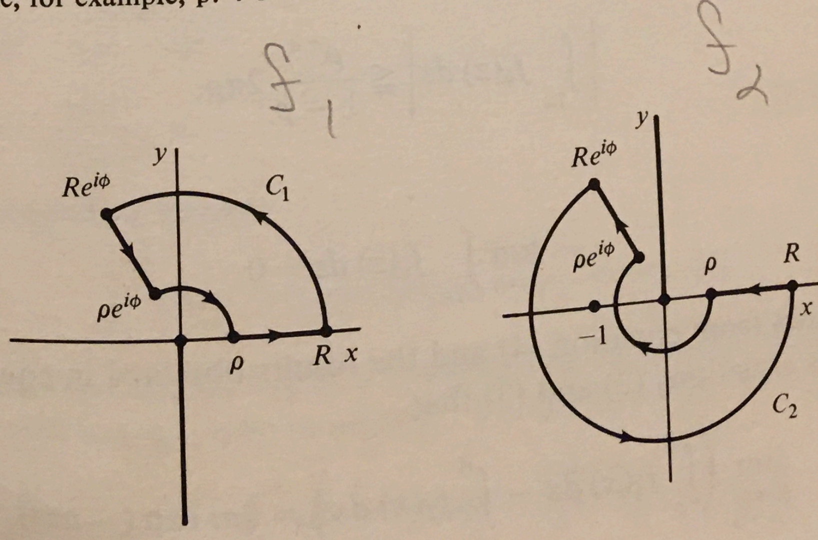

FIGURE 52

Thus

(3) $$\int_{C_2} f_2(z)\,dz = 2\pi i \exp(-a\pi i).$$

Since $f_1(z) = f_2(z)$ on the ray $\arg z = \phi$, it is also true that

(4) $$\int_{C_1} f_1(z)\,dz + \int_{C_2} f_2(z)\,dz = \int_\rho^R f_1(x)\,dx - \int_\rho^R f_2(x)\,dx$$
$$+ \int_{\Gamma_1} f_1(z)\,dz + \int_{\Gamma_2} f_2(z)\,dz + \int_{\gamma_1} f_1(z)\,dz + \int_{\gamma_2} f_2(z)\,dz$$

where Γ_k is the large circular arc and γ_k is the small circular arc of the simple closed contour C_k $(k = 1, 2)$ shown in Fig. 52.

When z is on Γ_k $(k = 1, 2)$,

$$|f_k(z)| = \left|\frac{z^{-a}}{z+1}\right| \leqq \frac{R^{-a}}{R-1};$$

and since the arc Γ_k is a portion of a circle whose circumference is $2\pi R$,

$$\left|\int_{\Gamma_k} f_k(z)\,dz\right| \leqq \frac{R^{-a}}{R-1} 2\pi R.$$

Hence

(5) $$\lim_{R\to\infty} \int_{\Gamma_k} f_k(z)\,dz = 0 \qquad (k = 1, 2).$$

When z is on γ_k $(k = 1, 2)$,

$$|f_k(z)| = \left|\frac{z^{-a}}{z+1}\right| \leqq \frac{\rho^{-a}}{1-\rho}.$$

Consequently,

$$\left|\int_{\gamma_k} f_k(z)\,dz\right| \leqq \frac{\rho^{-a}}{1-\rho} 2\pi\rho,$$

and

(6) $$\lim_{\rho\to 0} \int_{\gamma_k} f_k(z)\,dz = 0 \qquad (k = 1, 2).$$

It follows from equation (4) and the results obtained in equations (5) and (6) as well as equations (2) and (3) that

$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \left(\int_\rho^R f_1(x)\,dx - \int_\rho^R f_2(x)\,dx\right) = 2\pi i \exp(-a\pi i).$$

Since

$$\int_\rho^R f_1(x)\,dx - \int_\rho^R f_2(x)\,dx = \int_\rho^R \frac{1}{x+1}\,[e^{-a\,\mathrm{Log}\,x} - e^{-a(\mathrm{Log}\,x + 2\pi i)}]\,dx$$

$$= \int_\rho^R \frac{x^{-a}}{x+1}(1 - e^{-2\pi a i})\,dx,$$

we thus arrive at the result

$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \int_\rho^R \frac{x^{-a}}{x+1}\,dx = \frac{2\pi i \exp(-a\pi i)}{1 - \exp(-2a\pi i)}.$$

[Hint: Multiply numerator and denominator by exp(ia\pi)}]

That is,

$$\int_0^\infty \frac{x^{-a}}{x+1}\,dx = \frac{\pi}{\sin a\pi} \qquad (0 < a < 1).$$