Introduction: (Math 185A) Complex Variables or 1

"How to put the $i = \sqrt{-1}$ into Calculus"

Instructor: Blake Temple (Distinguished Professor)

Research: General Relativity and Shock Wave Theory

(2023) First nonlinear theory of time-periodic sound waves (w. Robin Young UMass)

(2020-23) Extend Kazden-DeTurck optimal regularity & Uhlenbeck compactness from Riemannian geometry to arbitrary connections (w Moritz Reintjes HK)

(2017-23) A theory of Dark Energy based on an Instability in the Standard Model of Cosmology (w Christopher Alexander and Zeke Vogler)

(2020-2023) A causal dissipative Relativistic Navier-Stokes eqn in which all shocks admit profiles (H Freistuehler, Konstanz)

Introduction to complex variables –

- the main goal of the class is to learn, and learn how to use, the Residue Theorem
- Rather than just learn to use it, I always wanted to know how one might discover the theory / how do you intuit the principles which underlie the theory? Partly this comes from understanding the proofs – maybe more from intuiting how one might "guess" the theory starting with what we already know. In this class we'll assume we know the Fundamental Theorem of Calculus FTC and the Math 210 theory of Line Integrals, and discover how, from this this, we can naturally, and very surprisingly, incorporate $i = \sqrt{-1}$ into Calculus!

- **History**: The subject of Complex Variables was written by

  Augustin Carchy 1787 – 1857

  Peter Dirchlet 1805 – 1859

  Karl Weierstrass 1815 – 1897

  * Georg Friedrich Bernard Riemann 1826 - 1866

"[Riemann... bound by his teacher Dirchlet...] would give acute, logical analyses of foundational questions, and would avoid long calculations as much as possible"

– Felix Klein

- **Motivation/Background**

By formally letting $i = \sqrt{-1}$, you can factor polynomials –

Eg $x^2 + 1 \neq 0$ has no real roots, and hence no real factors.

[Recall: if $P_n(r) = 0$, then $P_n(x) = (x-r)Q_{n-1}(x)$

$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$

$Q_{n-1}(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_0$

"roots & linear factors are the same"

So let $P(x) = x^2+1$

$r = \pm i = \pm\sqrt{-1}$ $\qquad (\pm i)^2+1 = i^2+1 = -1+1 = 0$

so $\quad x^2+1 = (x+i)(x-i)$ factors!

HW show if $P(r)=0$ then $(x-r)$ factors out
hint: Remainder Theorem

Turns out: We can prove that all polyn's with real coefficients uniquely factors into a product of linear and irreducible quadratic factors like x^2+1, x^2+x+1, etc

Conclude: if all quadratic polynomials can be factored using $i=\sqrt{-1}$, then all polynomials can be factored uniquely into a product of n linear factors using i

Quadratic Formula:

$$ax^2+bx+c=0 \qquad x_\pm = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$ax^2+bx+c = a(x-x_+)(x-x_-)$, $x_{\pm} = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

If $b^2 > 4ac$ then $x_{\pm}$ give real roots

If $b^2 < 4ac$ then $x_{\pm} = \dfrac{-b^2 \pm \sqrt{4ac-b^2}\, i}{2a}$

give two imaginary roots!

- So what is $i = \sqrt{-1}$?

General Complex Number: $z = \underbrace{x}_{\text{real part}} + i \underbrace{y}_{\text{imaginary part}}$

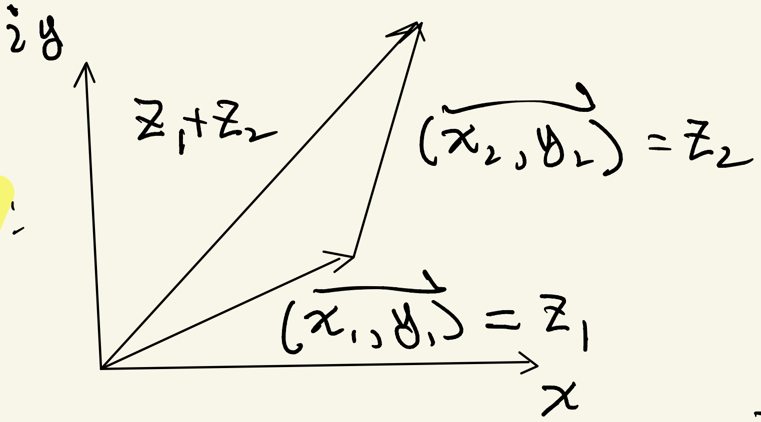
So really - complex numbers are just ordered pairs (x,y) corresponding to points in the plane $\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$

- adding and scalar multiplying (by real #) is the same as adding and scalar mult vectors in $\mathbb{R}^2$
- what $i = \sqrt{-1}$ gives you in addition to $\mathbb{R}^2$ is a way to multiply vectors!

I.e. as a vector space $\mathbb{C} \equiv$ complex plane $\equiv \mathbb{R}^2$ ⑥

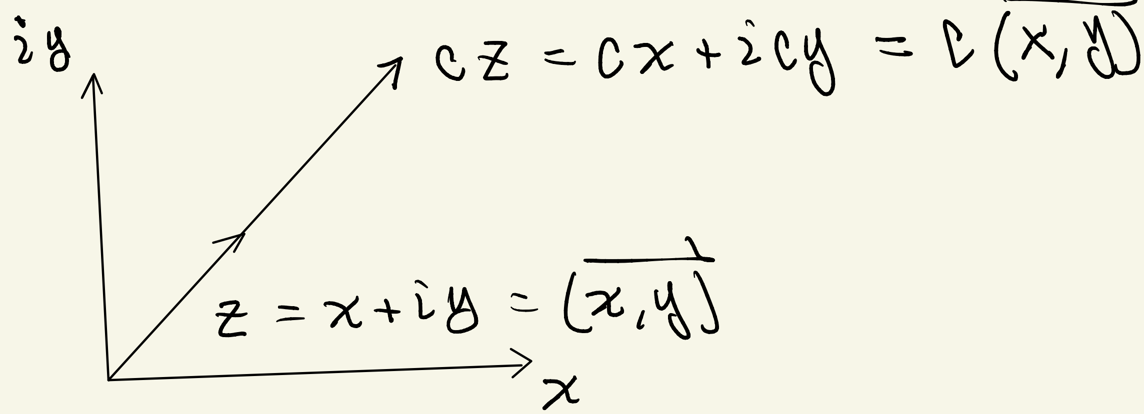
$$z = x + iy \equiv \overrightarrow{(x,y)}$$

Addition:



$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \equiv \overrightarrow{(x_1 + x_2, y_1 + y_2)}$$

Scalar multiplication by real number c



But $\mathbb{C}$ gives us a way to multiply ordered pairs as well:

$\mathbb{R}^2$ $\quad \overrightarrow{(x_1,y_1)} \cdot \overrightarrow{(x_2,y_2)} = ?$ not defined

$\mathbb{C}$ $\quad (x_1+iy_1)(x_2+iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

$$\equiv \overrightarrow{(x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)} \in \mathbb{R}^2$$

Conclude: $\mathbb{C} = \{x+iy : x,y \in \mathbb{R}\}$ is just the vector space $\mathbb{R}^2$ together with a way to multiply vectors.

In fact: This multiplication satisfies all the properties you need to do High School Algebra! These are called the Field Axioms

The Field Axioms organized in a table...

Defn: An addition $+$ and multiplication $\cdot$ defined on a set are said to define a Field if they satisfy –

Add	Mult
comm $z+w=w+z$	comm $z\cdot w = w\cdot z$
assoc $(z+w)+s = z+(w+s)$	assoc $(z\cdot w)\cdot s = z\cdot(w\cdot s)$
add id: $z+0=z$	mult id: $z\cdot 1 = z$
add inv: $(-z)+z=0$	mult inv $\left(\frac{1}{z}\right)z = 1$, $z\neq 0$

Distributive: $z\cdot(w+s) = z\cdot w + z\cdot s$

All Field Axioms are met with + & · defined by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 \cdot z_2 = (x_1 + iy_1)\cdot(x_2 + iy_2), \quad i^2 = -1$$

HW Check this is true.

Note: The deep part of the notion of Field is not checking each one (easy), but why are these, and only these, precisely the axioms you need to do Algebra ?!

Ex Prove $z \cdot 0 = 0$. (You put in the reasons)

Soln $z \cdot 0 = z \cdot (0 + 0) = z \cdot 0 + z \cdot 0$

So $z \cdot 0 - z \cdot 0 = z \cdot 0 + z \cdot 0 - z \cdot 0$

$0 = z \cdot 0 + 0 = z \cdot 0$ ✓

We say: $\mathbb{C}$ is a Field that extends $\mathbb{R}$:

Ie. $z = x + iy \in \mathbb{C}$ extends $x = x + i \cdot 0 \in \mathbb{R}$

"$\mathbb{R}$ is just $\mathbb{C}$ with zero imaginary part"

"$\mathbb{C}$ is just $\mathbb{R}^2$ together with a way to multiply"

HW: Find a formula for z^{-1} (Justify steps)

Soln $z^{-1} = \frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy}\,\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$

$$= \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2} = \frac{1}{x^2+y^2}(x-iy) = \frac{\bar z}{z\bar z}$$

Note the use of $\bar z = x - iy$; i.e., $\bar z \cdot z = x^2+y^2 = |z|^2$

Defn $\bar z = x - iy$ is the complex conjugate of $z = x+iy$

Basic Principle: "All formulas follow from the field axioms together with $i^2=-1$"

- What $\mathbb{R}$ has that $\mathbb{C}$ doesn't is an ordering

$x, y \in \mathbb{R} \Rightarrow x<y,\ x>y$ or $x=y$

* there is no such ordering of $\mathbb{R}^2$ or $\mathbb{C}$

Conclude: $\mathbb{C}$ is a Field which extends $\mathbb{R}$, and is just $\mathbb{R}^2$ together with a multiplication which turns $\mathbb{R}^2$ into a Field.

- To do Calculus, you need a notion of convergence. Turns out – using the notion of convergence in $\mathbb{R}^2$, we can put $i=\sqrt{-1}$ into Calculus as well!!