Sec. 1.1 and 1.2 – Homework Solutions

1.1 – (3) Let $z = a + bi$, so $z^2 = 3 - 4i$ iff $(a^2 - b^2) + 2abi = 3 - 4i$ iff $\underline{a^2 - b^2 = 3}$ and $\underline{2ab = -4}$.

Then $\underline{(a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2} = 3^2 + 4^2 = 25$, so $\underline{a^2 + b^2 = 5}$

Adding gives $2a^2 = 3 + 5 = 8$, so $a^2 = 4$ and $\underline{a = \pm 2}$. Since $b = -\frac{2}{a}$, $\underline{b = \mp 1}$;

so $\boxed{z = 2 - i}$ or $\boxed{z = -2 + i}$

<u>Remark</u> We could also solve this using the polar form

$3 - 4i = 5(\cos\theta + i\sin\theta)$ with $\theta = \tan^{-1}(-\frac{4}{3})$, but it would be harder.

(10) a) Let $z = x + yi$ and let $\phi_z : \mathbb{C} \to \mathbb{C}$ with $\phi_z(w) = zw$.

Then $\underline{\phi_z(1,0)} = z \cdot 1 = z = x + yi = \underline{(x, y)}$ ⟵ (first column)

and $\underline{\phi_z(0,1)} = z \cdot i = (x + yi)i = -y + xi = \underline{(-y, x)}$, ⟵ (second column)

so ϕ_z has the standard matrix $\left[\phi_z(\vec{e}_1) \mid \phi_z(\vec{e}_2)\right] = \boxed{\begin{bmatrix} x & -y \\ y & x \end{bmatrix}}$.

1.2 – (1) b) $z^4 + i = 0$ gives $z^4 = -i = 1\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$,

so $z = \sqrt[4]{1}\left(\cos\left(\frac{3\pi/2}{4} + \frac{2k\pi}{4}\right) + i\sin\left(\frac{3\pi/2}{4} + \frac{2k\pi}{4}\right)\right)$

$= \cos\left(\frac{3\pi}{8} + \frac{2k\pi}{4}\right) + i\sin\left(\frac{3\pi}{8} + \frac{2k\pi}{4}\right)$, $0 \le k \le 3$

Using the half-angle formulas

$\boxed{\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}}$ and $\boxed{\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}}$, we get

$\boxed{z_0 = \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2}i}$, $\boxed{z_1 = -\frac{\sqrt{2+\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2}i}$,

$\boxed{z_2 = -\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2}i}$, $\boxed{z_3 = \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2}i}$.

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(5) $\underline{(\cos x + i\sin x)^5 = \cos 5x + i\sin 5x}$ by De Moivre's Theorem, so

$(\cos x)^5 + 5(\cos x)^4(i\sin x) + 10(\cos x)^3(i\sin x)^2 + 10(\cos x)^2(i\sin x)^3 +$
$5(\cos x)(i\sin x)^4 + (i\sin x)^5 = \cos 5x + i\sin 5x$.

Then $\underline{(\cos^5 x - 10\cos^3 x\sin^2 x + 5\cos x\sin^4 x)} + i\,\underline{(5\cos^4 x\sin x - 10\cos^2 x\sin^3 x + \sin^5 x)} =$
$\underline{\cos 5x} + i\,\underline{\sin 5x}$,

so $\underline{\cos 5x = \cos^5 x - 10\cos^3 x\sin^2 x + 5\cos x\sin^4 x}$

and $\underline{\sin 5x = 5\cos^4 x\sin x - 10\cos^2 x\sin^3 x + \sin^5 x}$

MATH 185A

Sec. 1.2 and 1.3 — Solutions

1.2 – (9) Let $w^n = 1$ with $w \neq 1$.

Then $w^n - 1 = 0$, so $(w-1)(w^{n-1} + w^{n-2} + \dots + w + 1) = 0$ and therefore

$\underline{w^{n-1} + w^{n-2} + \dots + w + 1 = 0}$ since $w \neq 1$.

REMARK Notice that $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \dots + xy^{n-2} + y^{n-1})$, and it follows from this result that the sum of the nth roots of unity is 0.

(10) Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ where $a_0, \dots, a_n$ are real numbers.

If z_0 is a zero of $P(z)$, so $P(z_0) = 0$, then

$$\overline{P(z_0)} = \overline{a_n z_0^n + a_{n-1} z_0^{n-1} + \dots + a_1 z_0 + a_0} = \overline{0} = 0;$$

so $\overline{a_n}\,\overline{z_0}^n + \overline{a_{n-1}}\,\overline{z_0}^{n-1} + \dots + \overline{a_1}\,\overline{z_0} + \overline{a_0} = 0$. Since $a_i \in \mathbb{R}$, $\overline{a_i} = a_i$ for each i;

so $a_n \overline{z_0}^n + a_{n-1} \overline{z_0}^{n-1} + \dots + a_1 \overline{z_0} + a_0 = 0$.

Therefore $P(\overline{z_0}) = 0$, so $\overline{z_0}$ is also a zero of $P(z)$.

(15) Let $z = a + bi$.

Then $z^2 = (a^2 - b^2) + 2abi$ and $|z|^2 = a^2 + b^2$,

so $z^2 = |z|^2$ iff $\underline{a^2 - b^2 = a^2 + b^2 \text{ and } 2ab = 0}$

iff $2b^2 = 0$ and $2ab = 0$ iff $\underline{b = 0}$,

so $\underline{z^2 = |z|^2 \text{ iff } z \in \mathbb{R}}$.

1.3 – (1) b)
$$\sin(1+i) = \frac{e^{i(1+i)} - e^{-i(1+i)}}{2i} = \frac{e^{-1+i} - e^{1-i}}{2i}$$

$$= \frac{1}{2}(-i)\left[e^{-1}(\cos 1 + i\sin 1) - e(\cos 1 - i \sin 1)\right] \quad \left(\text{using } \frac{1}{i} = -i\right)$$

$$= \frac{1}{2}\left[e^{-1}(\sin 1 - i\cos 1) + e(\sin 1 + i\cos 1)\right]$$

$$= \boxed{\frac{1}{2}\sin 1\,(e^{-1} + e) + i\left(\frac{1}{2}\cos 1\right)(e - e^{-1})}$$

(2) b)
$$\cos(2+3i) = \frac{e^{i(2+3i)} + e^{-i(2+3i)}}{2} = \frac{1}{2}\left[e^{-3+2i} + e^{3-2i}\right]$$

$$= \frac{1}{2}\left[e^{-3}(\cos 2 + i\sin 2) + e^{3}(\cos 2 - i \sin 2)\right]$$

$$= \boxed{\frac{\cos 2}{2}(e^{-3} + e^{3}) + i\left(\frac{\sin 2}{2}\right)(e^{-3} - e^{3})}$$

REMARK We could also work 1b and 2b using the addition formulas (and the identities relating $\sin z$ and $\cos z$ to $\sinh z$ and $\cosh z$).