MATH 185A SOLUTIONS – SEC. 4.1

(1b) $\dfrac{1}{e^z - 1}$, $z_0 = 0$ $\operatorname{Res}(f;0) = \dfrac{g(0)}{h'(0)} = \dfrac{1}{e^0} = \boxed{1}$ (SINCE g HAS A ZERO OF ORDER 0, AND h HAS A ZERO OF ORDER 1, AT 0).

(1e) $\dfrac{e^z}{(z^2-1)^2}$, $z_0 = 1$ $\operatorname{Res}(f;1) = \dfrac{\phi'(1)}{1!}$ WHERE $\phi(z) = (z-1)^2 f(z) = \dfrac{e^z}{(z+1)^2}$, SINCE f HAS A POLE OF ORDER 2 AT $z_0 = 1$.

SINCE $\phi'(z) = \dfrac{(z+1)^2 e^z - e^z \cdot 2(z+1)}{(z+1)^4} = \dfrac{e^z(z-1)}{(z+1)^3}$, $\operatorname{Res}(f;1) = \phi'(1) = \boxed{0}$

(2a) $\dfrac{e^{z^2}}{z-1}$; $z_0 = 1$ $\lim\limits_{z\to 1} (z-1)\left(\dfrac{e^{z^2}}{z-1}\right) = \lim\limits_{z\to 1} e^{z^2} = e \neq 0$, SO $\operatorname{Res}(f;1) = \boxed{e}$

(2c) $\left(\dfrac{\cos z - 1}{z}\right)^2$, $z_0 = 0$ $\dfrac{\cos z - 1}{z} = \dfrac{\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) - 1}{z} = -\dfrac{z}{2!} + \dfrac{z^3}{4!} - \cdots$,

SO $\dfrac{\cos z - 1}{z}$ HAS A REMOVABLE SING. AT 0 AND THEREFORE $\left(\dfrac{\cos z - 1}{z}\right)^2$ DOES ALSO,

THUS $\operatorname{Res}(f;0) = \boxed{0}$

(8) a) $\dfrac{1}{e^z - 1}$ HAS SINGULAR POINTS WHERE $e^z = 1$: $\boxed{z = 2n\pi i, (n \in \mathbb{Z})}$

$\operatorname{Res}(f; 2n\pi i) = \dfrac{1}{e^z\big|_{2n\pi i}} = \dfrac{1}{1} = \boxed{1}$ (SINCE g HAS A ZERO OF ORDER 0 AT $2n\pi i$, AND h HAS A ZERO OF ORDER 1 THERE).

b) $\sin \frac{1}{z}$ SINCE $\sin z$ IS ENTIRE, $\boxed{z = 0}$ IS THE ONLY SINGULAR POINT,

SINCE $\sin\frac{1}{z} = \dfrac{1}{z} - \dfrac{\left(\frac{1}{z}\right)^3}{3!} + \dfrac{\left(\frac{1}{z}\right)^5}{5!} - \cdots = \dfrac{1}{z} - \dfrac{1}{6z^3} + \cdots$, $\operatorname{Res}(f;0) = \boxed{1}$

(9) $\dfrac{1}{z^2 \sin z}$; $z_0 = 0$ $z^2 \sin z = z^2\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots\right) = z^3 - \dfrac{z^5}{6} + \dfrac{z^7}{120} - \cdots$ SO

$$\begin{array}{r|l} & \frac{1}{z^3} + \frac{1}{6z} + \cdots \\ \hline z^3 - \frac{z^5}{6} + \cdots & 1 \\ & 1 - \frac{z^2}{6} + \cdots \\ \hline & \frac{z^2}{6} - \cdots \\ & \frac{z^2}{6} - \cdots \\ \hline \end{array}$$

SO $\operatorname{Res}(f;0) = \boxed{\frac{1}{6}}$

(OR, EQUIVALENTLY, SET $\left(\dfrac{b_3}{z^3} + \dfrac{b_2}{z^2} + \dfrac{b_1}{z} + a_0 + \cdots\right)\left(z^3 - \dfrac{z^5}{6} + \dfrac{z^7}{120} - \cdots\right) = 1$

AND SOLVE TO GET $b_3 = 1$ AND $b_1 - \frac{1}{6} b_3 = 0$, SO $b_1 = \boxed{\frac{1}{6}}$)

(10) IF $f_1(z) = \cdots + \dfrac{b_2}{(z-z_0)^2} + \dfrac{r_1}{z-z_0} + a_0 + \cdots$

AND $f_2(z) = \cdots + \dfrac{b_2^*}{(z-z_0)^2} + \dfrac{r_2}{z-z_0} + a_0^* + \cdots$, THEN

$(f_1 + f_2)(z) = \cdots + \dfrac{b_2 + b_2^*}{(z-z_0)^2} + \dfrac{r_1 + r_2}{z - z_0} + (a_0 + a_0^*) + \cdots$; SO $\operatorname{Res}(f_1 + f_2; z_0) = r_1 + r_2$.

OR use $\operatorname{Res}(f_1 + f_2; z_0) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma (f_1 + f_2) = \dfrac{1}{2\pi i}\int_\gamma f_1 + \dfrac{1}{2\pi i}\int_\gamma f_2 = \operatorname{Res}(f_1; z_0) + \operatorname{Res}(f_2; z_0)$

WHERE γ IS A SMALL CIRCLE AROUND z_0

MATH 185A SOLUTIONS – SEC. 4.1 AND 4.2

4.1 – (11) LET $f_1 = \dfrac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$

AND $f_2 = \dfrac{b_1^*}{z-z_0} + a_0^* + a_1^*(z-z_0) + \cdots$

THEN $f_1 f_2 = \dfrac{b_1 b_1^*}{(z-z_0)^2} + \dfrac{a_0 b_1^* + a_0^* b_1}{z-z_0} + (a_0 a_0^* + a_1 b_1^* + a_1^* b_1) + \cdots$,

SO $f_1 f_2$ HAS A POLE OF ORDER 2 AT z_0 AND $\operatorname{Res}(f_1 f_2; z_0) = a_0 b_1^* + a_0^* b_1$.

4.2 – (4) $\displaystyle\int_\gamma \frac{1}{e^z - 1}\,dz$ $\gamma: |z| = 9$ SING. INSIDE γ ARE $0, 2\pi i, -2\pi i$;

$$\int_\gamma \frac{1}{e^z-1}\,dz = 2\pi i\left[\operatorname{Res}(f;0) + \operatorname{Res}(f;2\pi i) + \operatorname{Res}(f;-2\pi i)\right] = 2\pi i\,[1+1+1] = \boxed{6\pi i}$$

(USING 4.1, #8A).

(6) $\displaystyle\int_\gamma \frac{5z-2}{z(z-1)}\,dz$ $\gamma: |z| = r$ WHERE $r > 1$

$$\int_\gamma \frac{5z-2}{z(z-1)}\,dz = 2\pi i\left[\operatorname{Res}(f;0) + \operatorname{Res}(f;1)\right] = 2\pi i\,[2+3] = 10\pi i$$

(SINCE $\displaystyle\lim_{z\to 0} \cancel{z}\cdot\frac{5z-2}{\cancel{z}(z-1)} = 2$ AND $\displaystyle\lim_{z\to 1} \cancel{(z-1)}\frac{5z-2}{z\cancel{(z-1)}} = 3$)

(9) a) $\displaystyle\int_{|z|=\frac{1}{2}} \frac{dz}{z(1-z)^3} = 2\pi i \operatorname{Res}(f;0) = 2\pi i\cdot 1 = \boxed{2\pi i}$ (SINCE $\displaystyle\lim_{z\to 0} \cancel{z}\cdot\frac{1}{\cancel{z}(1-z)^3} = 1$)

b) $\displaystyle\int_{|z|=\frac{1}{2}} \frac{e^z\,dz}{z(1-z)^3} = 2\pi i \operatorname{Res}(f;0) = 2\pi i\cdot 1 = \boxed{2\pi i}$ (SINCE $\displaystyle\lim_{z\to 0} \cancel{z}\cdot\frac{e^z}{\cancel{z}(1-z)^3} = \frac{1}{1} = 1$)

(10) c) $\displaystyle\int_{|z-1|=\frac{1}{2}} \frac{dz}{(1-z)^3} = 2\pi i \operatorname{Res}(f;1) = 2\pi i\cdot 0 = \boxed{0}$ SINCE $\dfrac{1}{(1-z)^3} = \dfrac{-1}{(z-1)^3}$, SO $b_1 = 0$,

(OR USE THE FUND. TH. FOR CONTOUR INTEGRALS, OR CAUCHY'S THEOREM, OR CAUCHY'S INTEGRAL FORMULA FOR DERIVATIVES)

(10) d) $\displaystyle\int_{|z-1|=\frac{1}{2}} \frac{e^z}{(1-z)^3}\,dz = 2\pi i \operatorname{Res}(f;1) = 2\pi i\left(-\frac{e}{2}\right) = \boxed{-e\pi i}$

SINCE $\dfrac{e^z}{(1-z)^3} = \dfrac{-e^z}{(z-1)^3}$, SO $\operatorname{Res}(f;1) = \dfrac{\phi''(1)}{2!} = -\dfrac{e}{2}$ WHERE $\phi(z) = -e^z$

OR $\displaystyle\int_\gamma \frac{e^z}{(1-z)^3}\,dz = \int_\gamma \frac{-e^z}{(z-1)^3}\,dz = \frac{2\pi i}{2!}\left(\frac{d^2}{dz^2}(-e^z)\Big|_{z=1}\right) = \frac{2\pi i}{2}\left(-e^z\big|_{z=1}\right)$

$= \boxed{-\pi e i}$

USING CAUCHY'S INTEGRAL FORMULA FOR DERIVATIVES.

(13) a) $\dfrac{(z-1)^3}{z(z+2)^3}$; $z_0 = \infty$

$\operatorname{Res}(f;\infty) = -\operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right);0\right)$ WHERE

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\cdot\frac{\left(\frac{1}{z}-1\right)^3}{\frac{1}{z}\left(\frac{1}{z}+2\right)^3} = \frac{(1-z)^3}{z(1+2z)^3},$$

so $\operatorname{Res}(f;\infty) = -\lim_{z\to 0} \dfrac{\cancel{z}(1-z)^3}{\cancel{z}(1+2z)^3} = -1$

b) $$\int_{|z|=3} \frac{(z-1)^3}{z(z+2)^3}\,dz = -2\pi i \operatorname{Res}(f;\infty) = -2\pi i\,(-1) = \boxed{2\pi i}$$

OR $$\int_{|z|=3} \frac{(z-1)^3}{z(z+2)^3}\,dz = 2\pi i\left[\operatorname{Res}(f;0) + \operatorname{Res}(f;-2)\right]$$

WHERE $\operatorname{Res}(f;0) = \lim_{z\to 0} \cancel{z}\cdot\dfrac{(z-1)^3}{\cancel{z}(z+2)^3} = -\dfrac{1}{8}$

AND $\operatorname{Res}(f;-2) = \dfrac{\phi''(-2)}{2!}$ WHERE $\phi(z) = \dfrac{(z-1)^3}{z}$, so

$$\phi(z) = \frac{z^3-3z^2+3z-1}{z} = z^2-3z+3-\frac{1}{z},$$

$$\phi'(z) = 2z-3+\frac{1}{z^2}, \quad \text{AND} \quad \phi''(z) = 2-\frac{2}{z^3}.$$

THEN $\operatorname{Res}(f;-2) = \dfrac{2-\frac{2}{-8}}{2!} = \dfrac{9/4}{2} = \dfrac{9}{8}$,

so $$\int_{|z|=3} \frac{(z-1)^3}{z(z+2)^3}\,dz = 2\pi i\left[-\frac{1}{8}+\frac{9}{8}\right] = 2\pi i\cdot 1 = \boxed{2\pi i}$$
