MATH 185A

SOLUTIONS - SEC. 4.3

① $\int_{-\infty}^{\infty} \frac{dx}{x^2-2x+4}$ LET $f(z) = \frac{1}{z^2-2z+4}$, SO $\deg Q \geq \deg P + 2$ AND

f HAS SINGULARITIES WHERE $z^2-2z+4=0$: $z^2-2z\underline{+1} = -4\underline{+1}$ $(z-1)^2=-3$ $z = 1 \pm \sqrt{3}i$;

$$\int_{-\infty}^{\infty} \frac{dx}{x^2-2x+4} = 2\pi i \operatorname{Res}(f; 1+\sqrt{3}i) = 2\pi i \left(\frac{1}{2(1+\sqrt{3}i)-2} \right) = \frac{2\pi i}{2\sqrt{3}i} = \boxed{\frac{\pi}{\sqrt{3}}}$$

② $$\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\frac{1}{2}(1-\cos 2x)}{x^2}\,dx = \frac{1}{4}\int_{-\infty}^{\infty} \frac{1-\cos 2x}{x^2}\,dx$$

LET $g(z) = \frac{1-e^{i2z}}{z^2}$ AND $f(z) = \frac{1}{z^2}$; SINCE $\deg(z^2) \geq \deg(1) + 2$,

$\int_{C_R} g(z)\,dz \to 0$ AS $R \to \infty$ (SINCE $g(z) = \frac{1}{z^2} - \frac{1}{z^2}\cdot e^{i2z}$);

SO $\int_{-\infty}^{\infty} \frac{1-e^{i2x}}{x^2}\,dx = \pi i \operatorname{Res}(g; 0) = \pi i(-2i) = 2\pi$

SINCE $g(z) = \frac{1-\left(1+2iz+\frac{(2iz)^2}{2!}+\cdots\right)}{z^2} = -\frac{2i}{z} - 2 + \cdots$.

THEREFORE $$\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{1}{4}\int_{-\infty}^{\infty} \frac{1-\cos 2x}{x^2}\,dx = \frac{1}{4}\operatorname{Re}\int_{-\infty}^{\infty} \frac{1-e^{i2x}}{x^2}\,dx = \frac{1}{4}(2\pi) = \boxed{\frac{\pi}{2}}$$

(OR USE $\operatorname{Res}(g; 0) = 2\cdot \left.\frac{\frac{d}{dz}(1-e^{i2z})}{\frac{d^2}{dz^2}(z^2)}\right|_{z=0} = \left.\frac{2\cdot(-2ie^{i2z})}{2}\right|_{z=0} = -2i$)

③ $\int_0^{\pi} \frac{1}{(a+b\cos\theta)^2}\,d\theta$, $0 < b < a$

$$\int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{1}{(a+b\cos\theta)^2}\,d\theta$$ (SINCE $\int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \int_{-\pi}^{0} \frac{d\theta}{(a+b\cos\theta)^2} = \int_{\pi}^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$)

($\cos\theta$ IS EVEN, AND HAS PERIOD 2π)

LET C BE $|z|=1$, AND $z=e^{i\theta}$ FOR $0 \leq \theta \leq 2\pi$; THIS GIVES

$$\frac{1}{2}\int_C \frac{1}{\left(a+b\left(\frac{z+\frac{1}{z}}{2}\right)\right)^2}\cdot\frac{1}{iz}\,dz = \frac{1}{2}\int_C \frac{1}{\left(a+b\left(\frac{z^2+1}{2z}\right)\right)^2}\cdot\frac{1}{iz}\,dz =$$

$$\frac{1}{2}\int_C \frac{4z^2}{(bz^2+2az+b)^2}\cdot\frac{1}{iz}\,dz = \underline{\frac{2}{i}\int_C \frac{z}{(bz^2+2az+b)^2}\,dz}$$

$bz^2+2az+b=0$ GIVES $z = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b} = \frac{-a \pm \sqrt{a^2-b^2}}{b}$,

SO LET $z_1 = \frac{-a+\sqrt{a^2-b^2}}{b}$ AND $z_2 = \frac{-a-\sqrt{a^2-b^2}}{b}$

SINCE $$z_1 = \frac{-a+\sqrt{a^2-b^2}}{b}\cdot\frac{-a-\sqrt{a^2-b^2}}{-a-\sqrt{a^2-b^2}} = \frac{a^2-(a^2-b^2)}{-b(a+\sqrt{a^2-b^2})} = \frac{b^2}{-b(a+\sqrt{a^2-b^2})} = \frac{-b}{a+\sqrt{a^2-b^2}},$$

$\underline{-1 < z_1 < 0}$ SINCE $b < a < a+\sqrt{a^2-b^2}$.

SINCE $z_2 = -\frac{a+\sqrt{a^2-b^2}}{b}$, $\underline{z_2 < -1}$.

THEREFORE $$\int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{2}{i}\left[2\pi i \operatorname{Res}\left(\frac{z}{(bz^2+2az+b)^2}; z_1\right)\right] = 4\pi \operatorname{Res}\left(\frac{z}{(bz^2+2az+b)^2}; z_1\right),$$

③ (CONTINUED)

LET $\phi(z) = (z-z_1)^2 f(z) = \dfrac{z}{b^2(z-z_2)^2}$ $\longleftarrow$ (SINCE $f(z) = \dfrac{z}{(b(z-z_1)(z-z_2))^2}$)

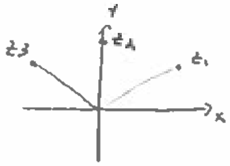
SO $$\phi'(z) = \frac{1}{b^2}\cdot\frac{(z-z_2)^2 - z\cdot 2(z-z_2)}{(z-z_2)^4} = \frac{1}{b^2}\cdot\frac{z-z_2-2z}{(z-z_2)^3} = -\frac{1}{b^2}\cdot\frac{z+z_2}{(z-z_2)^3}$$

AND $$\underline{\text{Res}(f;z_1)} = \phi'(z_1) = -\frac{1}{b^2}\cdot\frac{z_1+z_2}{(z_1-z_2)^3} = -\frac{1}{b^2}\cdot\frac{-2a/b}{\left(\frac{2\sqrt{a^2-b^2}}{b}\right)^3} = \frac{2a}{b^3}\cdot\frac{b^3}{8(a^2-b^2)^{3/2}} = \underline{\frac{a}{4(a^2-b^2)^{3/2}}}$$

THEREFORE $$\int_0^{\pi}\frac{d\theta}{(a+b\cos\theta)^2} = 4\pi\left(\frac{a}{4(a^2-b^2)^{3/2}}\right) = \boxed{\frac{\pi a}{(a^2-b^2)^{3/2}}}.$$

④ $$\int_0^{\infty}\frac{dx}{1+x^6} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{1+x^6}\,dx$$

LET $f(z) = \dfrac{1}{1+z^6}$, SO $\deg Q \geq \deg P + 2$ AND f HAS SIMPLE POLES WHERE $z^6 = -1$,
SO WHERE $z_k = e^{i\left(\frac{\pi}{6}+\frac{2k\pi}{6}\right)}$ FOR $k = 0, 1, \dots, 5$.



THEN $$\int_0^{\infty}\frac{dx}{1+x^6} = \frac{1}{2}\left(2\pi i\left[\text{Res}(f;z_1) + \text{Res}(f;z_2) + \text{Res}(f;z_3)\right]\right)$$
$$= \pi i\left[\text{Res}(f;e^{i\pi/6}) + \text{Res}(f;e^{i\pi/2}) + \text{Res}(f;e^{i5\pi/6})\right]$$

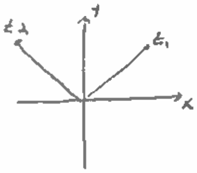
SINCE $\text{Res}(f;z_k) = \dfrac{1}{6z_k^5}$, THIS GIVES

$$\int_0^{\infty}\frac{dx}{1+x^6} = \pi i\left[\frac{1}{6e^{i5\pi/6}} + \frac{1}{6e^{i5\pi/2}} + \frac{1}{6e^{i25\pi/6}}\right] = \frac{\pi i}{6}\left[e^{-i5\pi/6} + e^{-i5\pi/2} + e^{-i25\pi/6}\right]$$
$$= \frac{\pi i}{6}\left[\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right) + (-i) + \left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)\right]$$
(SINCE $e^{-i5\pi/2} = e^{-i\pi/2} = -i$
AND $e^{-i25\pi/6} = e^{-i\pi/6}$)
$$= \frac{\pi i}{6}\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i - i + \frac{\sqrt{3}}{2} - \frac{1}{2}i\right] = \frac{\pi i}{6}\left[-2i\right] = \boxed{\frac{\pi}{3}}$$

⑤ $$\int_0^{\infty}\frac{\cos mx}{1+x^4}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos mx}{1+x^4}\,dx = \frac{1}{2}\,\text{Re}\int_{-\infty}^{\infty}\frac{e^{imx}}{1+x^4}\,dx$$

LET $f(z) = \dfrac{1}{1+z^4}$ AND $g(z) = \dfrac{e^{imz}}{1+z^4}$, SO $\deg Q \geq \deg P + 1$ (FOR $P(z) = 1$, $Q(z) = 1+z^4$)

AND $\displaystyle\int_{-\infty}^{\infty}\frac{e^{imx}}{1+x^4}\,dx = 2\pi i\left[\text{Res}(g;z_1) + \text{Res}(g;z_2)\right]$ WHERE $z_1 = e^{i\pi/4}$ AND $z_2 = e^{i3\pi/4}$



1) $$\text{Res}(g;z_1) = \frac{e^{imz_1}}{4z_1^3} = \frac{e^{im\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)}}{4e^{i3\pi/4}} = \frac{1}{4}e^{-m/\sqrt{2}}\,e^{i\left(\frac{m}{\sqrt{2}}-\frac{3\pi}{4}\right)}$$
$$= \frac{1}{4}e^{-m/\sqrt{2}}\left[\cos\left(\frac{m}{\sqrt{2}}-\frac{3\pi}{4}\right) + i\sin\left(\frac{m}{\sqrt{2}}-\frac{3\pi}{4}\right)\right]$$

2) $$\text{Res}(g;z_2) = \frac{e^{imz_2}}{4z_2^3} = \frac{e^{im\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)}}{4e^{i9\pi/4}} = \frac{1}{4}e^{-m/\sqrt{2}}\,e^{i\left(\frac{m}{\sqrt{2}}-\frac{\pi}{4}\right)}$$
$$= \frac{1}{4}e^{-m/\sqrt{2}}\left[\cos\left(\frac{m}{\sqrt{2}}+\frac{\pi}{4}\right) - i\sin\left(\frac{m}{\sqrt{2}}+\frac{\pi}{4}\right)\right]$$

SO $$\int_{-\infty}^{\infty}\frac{e^{imx}}{1+x^4}\,dx = 2\pi i\left(\frac{1}{4}e^{-m/\sqrt{2}}\right)\left[\cos\left(\frac{m}{\sqrt{2}}-\frac{3\pi}{4}\right) + \cos\left(\frac{m}{\sqrt{2}}+\frac{\pi}{4}\right) + i\left(\sin\left(\frac{m}{\sqrt{2}}-\frac{3\pi}{4}\right) - \sin\left(\frac{m}{\sqrt{2}}+\frac{\pi}{4}\right)\right)\right]$$

⑤ (CONTINUED)

$$\text{THEN} \int_0^\infty \frac{\cos mx}{1+x^4}\,dx = \frac{1}{2}\,\mathrm{Re}\int_{-\infty}^{\infty}\frac{e^{imx}}{1+x^4}\,dx = \frac{\pi}{4}e^{-m/\sqrt{2}}\left(\sin\left(\frac{m}{\sqrt{2}}+\frac{\pi}{4}\right)-\sin\left(\frac{m}{\sqrt{2}}-\frac{3\pi}{4}\right)\right)$$

$$= \frac{\pi}{4}e^{-m/\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(\sin\frac{m}{\sqrt{2}}+\cos\frac{m}{\sqrt{2}}\right)-\left(-\frac{1}{\sqrt{2}}\right)\left(\sin\frac{m}{\sqrt{2}}+\cos\frac{m}{\sqrt{2}}\right)\right] = \boxed{\frac{\pi}{2\sqrt{2}}e^{-m/\sqrt{2}}\left(\sin\frac{m}{\sqrt{2}}+\cos\frac{m}{\sqrt{2}}\right)}$$

(FOR $m>0$)

⑦

$$\int_0^\infty \frac{x\sin x}{1+x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x\sin x}{1+x^2}\,dx = \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty}\frac{xe^{ix}}{1+x^2}\,dx$$

LET $f(z) = \dfrac{z}{1+z^2}$, SO $\deg Q \geq \deg P + 1$, AND LET $g(z) = \dfrac{ze^{iz}}{1+z^2}$.

$$\text{THEN} \int_{-\infty}^{\infty}\frac{xe^{ix}}{1+x^2}\,dx = 2\pi i\,\mathrm{Res}(g;i) \quad (\text{SINCE } f \text{ HAS SINGULARITIES AT } \pm i)$$

$$= 2\pi i \lim_{z\to i}\frac{ze^{iz}}{z+i} = 2\pi i\cdot\frac{ie^{-1}}{2i} = \frac{\pi}{e}i,$$

$$\text{SO} \int_0^\infty \frac{x\sin x}{1+x^2}\,dx = \frac{1}{2}\,\mathrm{Im}\left(\frac{\pi}{e}i\right) = \boxed{\frac{\pi}{2e}}$$

⑨

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{dx}{(x-a)^2(x-1)} \quad (\text{WHERE } \mathrm{Im}\, a>0)$$

LET $f(z) = \dfrac{1}{(z-a)^2(z-1)}$, SO $\deg(Q) \geq \deg(P) + 2$

$$\text{THEN P.V.}\int_{-\infty}^{\infty}\frac{dx}{(x-a)^2(x-1)} = \underline{2\pi i\,\mathrm{Res}(f;a) + \pi i\,\mathrm{Res}(f;1)}$$

(SINCE f HAS A SING. AT a IN H AND A SIMPLE POLE AT 1 ON THE X-AXIS).

1) $\mathrm{Res}(f;a) = \phi'(a)$ WHERE $\phi(z) = (z-a)^2 f(z) = \dfrac{1}{z-1}$,

$$\text{SO } \mathrm{Res}(f;a) = -\frac{1}{(z-1)^2}\bigg|_{z=a} = -\frac{1}{(a-1)^2}$$

2) $\mathrm{Res}(f;1) = \displaystyle\lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{1}{(z-a)^2} = \frac{1}{(1-a)^2} = \frac{1}{(a-1)^2}$

$$\text{THUS P.V.}\int_{-\infty}^{\infty}\frac{dx}{(x-a)^2(x-1)} = 2\pi i\left(-\frac{1}{(a-1)^2}\right)+\pi i\left(\frac{1}{(a-1)^2}\right) = \boxed{\frac{-\pi i}{(a-1)^2}}$$

⑰

$$\int_{-\infty}^{\infty}\frac{\cos bx}{x^2+a^2}\,dx = \mathrm{Re}\int_{-\infty}^{\infty}\frac{e^{ibx}}{x^2+a^2}\,dx$$

LET $f(z) = \dfrac{1}{z^2+a^2}$, SO $\deg Q \geq \deg P + 1$, AND LET $g(z) = \dfrac{e^{ibz}}{z^2+a^2}$.

$$\text{THEN} \int_{-\infty}^{\infty}\frac{e^{ibx}}{x^2+a^2}\,dx = 2\pi i\,\mathrm{Res}(g;ai) \quad \text{SINCE } f \text{ HAS SINGULARITIES AT } \pm ai,$$

$$\text{AND } \underline{\mathrm{Res}(g;ai)} = \lim_{z\to ai}\frac{e^{ibz}}{z+ai} = \underline{\frac{e^{-ab}}{2ai}}$$

$$\text{THEREFORE} \int_{-\infty}^{\infty}\frac{e^{ibx}}{x^2+a^2}\,dx = 2\pi i\left(\frac{e^{-ab}}{2ai}\right) = \frac{\pi}{ae^{ab}}, \quad \text{SO} \int_{-\infty}^{\infty}\frac{\cos bx}{x^2+a^2}\,dx = \boxed{\frac{\pi}{ae^{ab}}}$$

(WHERE $a, b > 0$)

MATH 185A SOLUTIONS – SEC. 4.3 AND 4.4

4.3 – ⑳a $\displaystyle\int_0^{\pi} \frac{1}{1+\sin^2\theta}\,d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1}{1+\sin^2\theta}\,d\theta$ SINCE $\displaystyle\int_0^{\pi} \frac{d\theta}{1+\sin^2\theta} = \int_{-\pi}^{0} \frac{d\theta}{1+\sin^2\theta} = \int_{\pi}^{2\pi} \frac{d\theta}{1+\sin^2\theta}$

(BECAUSE $\sin^2\theta$ IS EVEN, AND HAS PERIOD 2π)

LETTING C BE $|z|=1$ WITH $z=e^{i\theta}$ GIVES

$$\frac{1}{2}\int_C \frac{1}{1+\left(\frac{z-\frac{1}{z}}{2i}\right)^2}\cdot\frac{1}{iz}\,dz = \frac{1}{2}\int_C \frac{1}{1+\left(\frac{z^2-1}{2iz}\right)^2}\cdot\frac{1}{iz}\,dz = \frac{1}{2}\int_C \frac{1}{1+\frac{(z^2-1)^2}{-4z^2}}\cdot\frac{1}{iz}\,dz$$

$$= \frac{1}{2}\int_C \frac{-4z^2}{(z^2-1)^2-4z^2}\cdot\frac{1}{iz}\,dz = -\frac{2}{i}\int_C \frac{z}{(z^2-1-2z)(z^2-1+2z)}\,dz$$

IF $f(z) = \dfrac{z}{(z^2-2z-1)(z^2+2z-1)}$, f HAS SING. AT $z = 1 \pm \sqrt{2}$ AND $z = -1 \pm \sqrt{2}$; SO $\underline{z_1 = 1-\sqrt{2}}$ AND $\underline{z_2 = -1+\sqrt{2}}$ ARE THE ONLY SING. INSIDE C.

$$\text{THEN } \int_0^{\pi} \frac{1}{1+\sin^2\theta}\,d\theta = -\frac{2}{i}\cdot 2\pi i\left[\text{Res}(f;z_1) + \text{Res}(f;z_2)\right]$$

$$= -4\pi\left[\text{Res}(f;z_1) + \text{Res}(f;z_2)\right]$$

$$\text{WHERE } \text{Res}(f;z_k) = \frac{z_k}{\frac{d}{dz}(z^4-6z^2+1)\big|_{z=z_k}} = \frac{z_k}{4z_k^3-12z_k} = \frac{1}{4(z_k^2-3)} = \frac{1}{4(3-2\sqrt{2}-3)} = \frac{1}{4(-2\sqrt{2})},$$

$$\text{SO } \int_0^{\pi} \frac{1}{1+\sin^2\theta}\,d\theta = -4\pi\left[\frac{1}{-8\sqrt{2}} + \frac{1}{-8\sqrt{2}}\right] = -4\pi\left[\frac{2}{-8\sqrt{2}}\right] = \boxed{\frac{\pi}{\sqrt{2}}}$$

4.4 – ① $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$ LET $f(z) = \dfrac{1}{z^4}$, SO $\deg Q \geq \deg P + 2$; AND f HAS A SING. AT 0:

$$\lim_{N\to\infty}\left[\sum_{n=-N}^{-1}\frac{1}{n^4} + \sum_{n=1}^{N}\frac{1}{n^4}\right] = -\text{Res}\left(\frac{\pi\cot\pi z}{z^4};0\right)$$

$$\text{SO } \lim_{N\to\infty} 2\sum_{n=1}^{N}\frac{1}{n^4} = -\text{Res}\left(\frac{\pi\cot\pi z}{z^4};0\right)$$

$$\text{AND } \sum_{n=1}^{\infty}\frac{1}{n^4} = -\frac{1}{2}\,\text{Res}\left(\frac{\pi\cot\pi z}{z^4};0\right)$$

$$\cot z = \frac{\cos z}{\sin z} = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \cdots \longleftarrow$$

$$\begin{array}{r|l} & \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \cdots \\ \hline z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots & 1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots \\ & 1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots \\ \hline & -\frac{z^2}{3} + \frac{z^4}{30} - \cdots \\ & -\frac{z^2}{3} + \frac{z^4}{18} - \cdots \\ \hline & -\frac{z^4}{45} + \cdots \end{array}$$

$$\pi\cot\pi z = \pi\left[\frac{1}{\pi z} - \frac{\pi z}{3} - \frac{\pi^3 z^3}{45} - \cdots\right]$$

$$= \frac{1}{z} - \frac{\pi^2}{3}z - \frac{\pi^4}{45}z^3 - \cdots$$

$$\frac{\pi\cot\pi z}{z^4} = \frac{1}{z^5} - \frac{\pi^2}{3}\cdot\frac{1}{z^3} - \frac{\pi^4}{45}\cdot\frac{1}{z} - \cdots$$

$$\text{SO } \text{Res}\left(\frac{\pi\cot\pi z}{z^4};0\right) = -\frac{\pi^4}{45}$$

$$\text{AND } \sum_{n=1}^{\infty}\frac{1}{n^4} = -\frac{1}{2}\left(-\frac{\pi^4}{45}\right) = \boxed{\frac{\pi^4}{90}}$$

MATH 185A SOLUTIONS – SEC. 4.3

(20b) $$\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)^2}\,dx$$

LET $f(z) = \dfrac{z^2}{(z^2+a^2)^2}$, SO f HAS SING. AT $z = \pm ai$ AND $\deg Q \geq \deg P + 2$.

THEN $\displaystyle\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\,dx = \frac{1}{2}\left(2\pi i \operatorname{Res}(f; ai)\right) = \pi i\, \phi'(ai)$ WHERE

$$\phi(z) = (z-ai)^2 f(z) = \frac{z^2}{(z+ai)^2} = \left(\frac{z}{z+ai}\right)^2, \quad \text{SO}$$

$$\phi'(z) = 2\left(\frac{z}{z+ai}\right)\cdot\frac{ai}{(z+ai)^2} = \frac{2z(ai)}{(z+ai)^3}.$$

THEREFORE $\displaystyle\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\,dx = \pi i\left(\frac{2ai\,(ai)}{(2ai)^3}\right) = \pi i\left(\frac{-2a^2}{-8a^3 i}\right) = \boxed{\frac{\pi}{4a}}$

(20c) $$\int_{-\infty}^{\infty}\frac{x^3\sin x}{(x^2+1)^2}\,dx = \operatorname{Im}\int_{-\infty}^{\infty}\frac{x^3 e^{ix}}{(x^2+1)^2}\,dx$$

LET $f(z) = \dfrac{z^3}{(z^2+1)^2}$, SO $\deg Q \geq \deg P + 1$, AND LET $g(z) = \dfrac{z^3 e^{iz}}{(z^2+1)^2}$.

THEN $\displaystyle\int_{-\infty}^{\infty}\frac{x^3 e^{ix}}{(x^2+1)^2}\,dx = 2\pi i \operatorname{Res}(g; i)$ (SINCE f HAS SING. AT $\pm i$).

LET $\phi(z) = (z-i)^2 g(z) = (z-i)^2\cdot\dfrac{z^3 e^{iz}}{((z-i)(z+i))^2} = \dfrac{z^3 e^{iz}}{(z+i)^2}$.

THEN $$\phi'(z) = \frac{(z+i)^2\left[z^3\cdot e^{iz}\cdot i + 3z^2 e^{iz}\right] - z^3 e^{iz}\cdot 2(z+i)}{(z+i)^4}$$

$$= \frac{\cancel{(z+i)}\, z^2 e^{iz}\left[iz(z+i) + 3(z+i) - 2z\right]}{(z+i)^{\cancel{4}\,3}} = \frac{z^2 e^{iz}\cdot i\,(z^2+3)}{(z+i)^3}$$

SO $\operatorname{Res}(g; i) = \phi'(i) = \dfrac{-e^{-1}(2i)}{-8i} = \dfrac{1}{4e}$.

THUS $\displaystyle\int_{-\infty}^{\infty}\frac{x^3 e^{ix}}{(x^2+1)^2}\,dx = 2\pi i\cdot\frac{1}{4e} = \frac{\pi}{2e}\cdot i$,

SO $\displaystyle\int_{-\infty}^{\infty}\frac{x^3\sin x}{(x^2+1)^2}\,dx = \operatorname{Im}\left(\frac{\pi}{2e}\cdot i\right) = \boxed{\frac{\pi}{2e}}$