MATH 185A SOLUTIONS – SEC. 1.4

(2b) 1) $\lim_{z \to z_0} f(z) = \lim_{(x,y)\to(x_0,y_0)} u(x,y) + i \lim_{(x,y)\to(x_0,y_0)} v(x,y)$ IF BOTH LIMITS ON THE RIGHT EXIST.

PF LET $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = a$ AND $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = b$, AND LET $\varepsilon > 0$.

THERE IS A $\delta_1 > 0$ SUCH THAT $|u(x,y) - a| < \frac{\varepsilon}{2}$ IF $0 < |(x,y) - (x_0,y_0)| < \delta_1$, AND THERE IS A $\delta_2 > 0$ SUCH THAT $|v(x,y) - b| < \frac{\varepsilon}{2}$ IF $0 < |(x,y) - (x_0,y_0)| < \delta_2$.

IF $\delta = \min\{\delta_1, \delta_2\}$, THEN $0 < |z - z_0| < \delta \implies$ IF $w = a + bi$,

$$|f(z) - w| \le |Re(f(z) - w)| + |Im(f(z) - w)| = |u(x,y) - a| + |v(x,y) - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

THEREFORE $\lim_{z \to z_0} f(z) = w = a + bi$.

2) IF $\lim_{z \to z_0} f(z) = w = a + bi$, THEN $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = a$ AND $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = b$.

PF LET $\varepsilon > 0$. SINCE $\lim_{z \to z_0} f(z) = w$, THERE IS A $\delta > 0$ SUCH THAT IF $0 < |z - z_0| < \delta$, THEN $|f(z) - w| < \varepsilon$.

THEN $0 < |z - z_0| < \delta \implies |u(x,y) - a| = |Re(f(z) - w)| \le |f(z) - w| < \varepsilon$ AND

$|v(x,y) - b| = |Im(f(z) - w)| \le |f(z) - w| < \varepsilon$,

SO $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = a$ AND $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = b$.

3) f IS CONTINUOUS AT z_0 IFF u AND v ARE CONTINUOUS AT $(x_0, y_0) = z_0$.

PF USING PARTS 1) AND 2), f IS CONTINUOUS AT z_0 IFF $\lim_{z \to z_0} f(z) = f(z_0)$

IFF $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u(x_0,y_0)$ AND $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v(x_0,y_0)$

IFF u AND v ARE CONTINUOUS AT (x_0, y_0).

(11) LET $r = |z|$.

1) IF $r < 1$, THEN $\underline{\lim_{n\to\infty} n z^n = 0}$ SINCE IF $s = \frac{1}{r} > 1$, THEN

$$\lim_{n\to\infty} |n z^n| = \lim_{n\to\infty} n r^n = \lim_{n\to\infty} \frac{n}{s^n} = \lim_{n\to\infty} \frac{1}{s^n \ln s} = 0$$ USING L'HOSPITAL'S RULE.

2) IF $r \ge 1$, THEN $\underline{\lim_{n\to\infty} n z^n \text{ DOES NOT EXIST}}$ SINCE $\lim_{n\to\infty} |n z^n| = \lim_{n\to\infty} n r^n = \infty$

(BECAUSE $n r^n \ge n$ AND $\lim_{n\to\infty} n = \infty$).

THEREFORE $\underline{\lim_{n\to\infty} n z^n \text{ EXISTS IFF } |z| < 1}$.

(23) IF $|z| > 1$, THEN $\lim_{n\to\infty} \frac{z^n}{n} = \infty$.

PF IF $r = |z|$, THEN $\lim_{n\to\infty} \left|\frac{z^n}{n}\right| = \lim_{n\to\infty} \frac{r^n}{n} = \lim_{n\to\infty} \frac{r^n \ln r}{1} = \infty$ USING L'HOSPITAL'S RULE,

SO $\lim_{n\to\infty} \frac{z^n}{n} = \infty$.

(OR USE $r = 1 + c$ WITH $c > 0$, SO $r^n = (1+c)^n = 1 + nc + \frac{n(n-1)}{2!} c^2 + \dots$

AND THEREFORE $\frac{r^n}{n} > \frac{\frac{n(n-1)}{2} c^2}{n} = \frac{(n-1)c^2}{2}$ FOR $n \ge 2$, SO $\frac{r^n}{n} \to \infty$)

1.4 – 18 Show that $f: A \to \mathbb{C}$ is continuous iff $z_n \to z_0$ in A implies that $f(z_n) \to f(z_0)$.

PF $\Longrightarrow$ Suppose that f is continuous on A, and let $z_n \to z_0$ in A.

1) If $\varepsilon > 0$ is given, then there is a $\delta > 0$ such that if $|z - z_0| < \delta$, then $|f(z) - f(z_0)| < \varepsilon$ since f is continuous at z_0.

2) Since $z_n \to z_0$, there is an $N \in \mathbb{N}$ such that $n \geq N \Longrightarrow |z_n - z_0| < \delta$.

Therefore $n \geq N \Longrightarrow |f(z_n) - f(z_0)| < \varepsilon$, so $f(z_n) \to f(z_0)$.

$\Longleftarrow$ (using the contrapositive)

Assume that f is not continuous at a point z_0 in A.
Then there is an $\varepsilon > 0$ such that for every $\delta > 0$,
there is a $z \in A$ with $|z - z_0| < \delta$ and $|f(z) - f(z_0)| \geq \varepsilon$.
In particular, for each $n \in \mathbb{N}$ there is a $z_n \in A$ with
$|z_n - z_0| < \frac{1}{n}$ and $|f(z_n) - f(z_0)| \geq \varepsilon$. (taking $\delta = \frac{1}{n}$)
Therefore $z_n \to z_0$, but $f(z_n) \not\to f(z_0)$.

CH. 1 RE – 4 If $z = a + bi$, $z^2 = 1 + i$ iff $(a^2 - b^2) + 2abi = 1 + i$ iff $\underline{a^2 - b^2 = 1}$ and $\underline{2ab = 1}$.
Then $(a^2 + b^2)^2 = (a^2 - b^2)^2 + (2ab)^2 = 1^2 + 1^2 = 2$, so $\underline{a^2 + b^2 = \sqrt{2}}$.
Therefore $2a^2 = 1 + \sqrt{2}$, so $a^2 = \frac{1+\sqrt{2}}{2}$ and $\underline{a = \pm\sqrt{\frac{1+\sqrt{2}}{2}}}$; and
$2b^2 = \sqrt{2} - 1$, so $b^2 = \frac{\sqrt{2}-1}{2}$ and $\underline{b = \pm\sqrt{\frac{\sqrt{2}-1}{2}}}$ (where $ab = \frac{1}{2} > 0$).

Thus $\boxed{z = \sqrt{\frac{1+\sqrt{2}}{2}} + \sqrt{\frac{\sqrt{2}-1}{2}}\, i}$ and $\boxed{z = -\sqrt{\frac{1+\sqrt{2}}{2}} - \sqrt{\frac{\sqrt{2}-1}{2}}\, i}$

OR $z^2 = 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, so

$$z = \sqrt{\sqrt{2}}\left(\cos\left(\frac{\pi/4}{2} + \frac{2k\pi}{2}\right) + i\left(\sin\left(\frac{\pi/4}{2} + \frac{2k\pi}{2}\right)\right)\right), \quad k = 0, 1$$

so $z_0 = \sqrt[4]{2}\left(\left(\cos\frac{\pi}{8}\right) + i\sin\frac{\pi}{8}\right) = \sqrt[4]{2}\left(\sqrt{\frac{1+\sqrt{2}/2}{2}} + i\sqrt{\frac{1-\sqrt{2}/2}{2}}\right)$ $\longleftarrow$ $\boxed{\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \quad \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}}$

$$= \boxed{\sqrt[4]{2}\left(\frac{\sqrt{2+\sqrt{2}}}{2} + i\,\frac{\sqrt{2-\sqrt{2}}}{2}\right)}$$

or $z_1 = \sqrt[4]{2}\left(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right) = \boxed{-\sqrt[4]{2}\left(\frac{\sqrt{2+\sqrt{2}}}{2} + i\,\frac{\sqrt{2-\sqrt{2}}}{2}\right)}$

REMARK We can also write this as

$$\boxed{z = \pm 2^{-3/4}\left(\sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}}\right)}$$

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(2) Let $z = re^{i\theta}$ where $-\pi < \theta \le \pi$, so $z^2 = r^2 e^{i2\theta}$.

Then $\operatorname{Log} z = \ln r + i\theta$, and $\operatorname{Log} z^2 = \ln r^2 + i\theta'$, where $-\pi < \theta' \le \pi$ and $\theta' = 2\theta \pmod{2\pi}$.

Then $\operatorname{Log} z^2 = 2\operatorname{Log} z$ iff $\ln r^2 + i\theta' = 2\ln r + 2i\theta$ iff $\theta' = 2\theta$ iff $-\pi < 2\theta \le \pi$

iff $-\frac{\pi}{2} < \theta \le \frac{\pi}{2}$. Therefore $\operatorname{Log} z^2 = 2\operatorname{Log} z$ iff $-\frac{\pi}{2} < \operatorname{Arg} z \le \frac{\pi}{2}$ iff

z is in the right half-plane or z is on the positive half of the imaginary axis.

(6) $\sin z = \sqrt{3}$, so $\cos^2 z = 1 - \sin^2 z = 1 - 3 = -2$ and $\cos z = \pm\sqrt{2}\, i$.

Then $e^{iz} = \cos z + i\sin z = \pm\sqrt{2}\, i + \sqrt{3}\, i = (\sqrt{3} \pm \sqrt{2})\, i$,

so $iz = \operatorname{Log}\left((\sqrt{3} \pm \sqrt{2})\, i\right) = \ln(\sqrt{3} \pm \sqrt{2}) + i\left(\frac{\pi}{2} + 2n\pi\right)$

and $z = -i\operatorname{Log}\left((\sqrt{3} \pm \sqrt{2})\, i\right) = \boxed{\frac{\pi}{2} + 2n\pi - i\ln(\sqrt{3} \pm \sqrt{2})}$, where $n \in \mathbb{Z}$.

OR $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} = \sqrt{3}$, so $e^{iz} - e^{-iz} = 2\sqrt{3}\, i$ and therefore

$e^{iz} - 2\sqrt{3}\, i - e^{-iz} = 0$. Then $e^{2iz} - 2\sqrt{3}\, i\, e^{iz} - 1 = 0$, so

$e^{iz} = \dfrac{2\sqrt{3}\, i \pm \sqrt{-12 - (-4)}}{2} = \dfrac{2\sqrt{3}\, i \pm \sqrt{-8}}{2} = \dfrac{2\sqrt{3}\, i \pm 2\sqrt{2}\, i}{2} = (\sqrt{3} \pm \sqrt{2})\, i$.

Then $iz = \operatorname{Log}\left((\sqrt{3} \pm \sqrt{2})\, i\right) = \ln(\sqrt{3} \pm \sqrt{2}) + i\left(\frac{\pi}{2} + 2n\pi\right)$,

so $z = -i\left(\ln(\sqrt{3} \pm \sqrt{2}) + i\left(\frac{\pi}{2} + 2n\pi\right)\right) = \boxed{\frac{\pi}{2} + 2n\pi - i\ln(\sqrt{3} \pm \sqrt{2})}$, where $n \in \mathbb{Z}$.

(8b) $|z - 1| = 2|z|$. If we let $z = x + yi$, $z - 1 = (x-1) + yi$, so

$|z-1|^2 = 4|z|^2$ iff $(x-1)^2 + y^2 = 4(x^2 + y^2)$ iff

$x^2 - 2x + 1 + y^2 = 4x^2 + 4y^2$ iff $3x^2 + 3y^2 + 2x = 1$ iff

$x^2 + y^2 + \frac{2}{3}x = \frac{1}{3}$ iff

$\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) + y^2 = \frac{1}{3} + \frac{1}{9}$ iff

$\left(x + \frac{1}{3}\right)^2 + y^2 = \frac{4}{9}$

This is $\boxed{\text{the circle with center } \left(-\tfrac{1}{3}, 0\right) \text{ and radius } r = \tfrac{2}{3}}$.

REMARK We can also write the equation of the circle

as $\left|z + \frac{1}{3}\right| = \frac{2}{3}$.