MATH 185A SOLUTIONS – Sec. 1.5

(4) If $\gamma:(a,b)\to\mathbb{C}$ is differentiable and $f:A\to\mathbb{C}$ is analytic with $\gamma([a,b])\subseteq A$, then $\sigma=f\circ\gamma$ is differentiable with $\sigma'(t)=f'(\gamma(t))\gamma'(t)$.

PF Let $t_1\in(a,b)$, and define $h:A\to\mathbb{C}$ by $h(w)=\begin{cases}\dfrac{f(w)-f(\gamma(t_1))}{w-\gamma(t_1)}, & \text{if } w\neq\gamma(t_1)\\ f'(\gamma(t_1)), & \text{if } w=\gamma(t_1).\end{cases}$

Then h is continuous at $\gamma(t_1)$ since

$\lim_{w\to\gamma(t_1)} h(w)=f'(\gamma(t_1))=h(\gamma(t_1))$, so $h\circ\gamma$ is continuous at t_1.

Then $\underline{\dfrac{\sigma(t)-\sigma(t_1)}{t-t_1}=h(\gamma(t))\cdot\dfrac{\gamma(t)-\gamma(t_1)}{t-t_1}}$ for $t\neq t_1$ since

i) $\dfrac{\sigma(t)-\sigma(t_1)}{t-t_1}=\dfrac{f(\gamma(t))-f(\gamma(t_1))}{\gamma(t)-\gamma(t_1)}\cdot\dfrac{\gamma(t)-\gamma(t_1)}{t-t_1}=h(\gamma(t))\cdot\dfrac{\gamma(t)-\gamma(t_1)}{t-t_1}$ if $\gamma(t)\neq\gamma(t_1)$,

and ii) both sides are zero if $\gamma(t)=\gamma(t_1)$;

so $\underline{\sigma'(t_1)}=\lim_{t\to t_1}\dfrac{\sigma(t)-\sigma(t_1)}{t-t_1}=\lim_{t\to t_1}h(\gamma(t))\cdot\dfrac{\gamma(t)-\gamma(t_1)}{t-t_1}$

$=\left(\lim_{t\to t_1}h(\gamma(t))\right)\left(\lim_{t\to t_1}\dfrac{\gamma(t)-\gamma(t_1)}{t-t_1}\right)$

$=h(\gamma(t_1))\cdot\gamma'(t_1)=\underline{f'(\gamma(t_1))\gamma'(t_1)}.$

(5) b) $f(z)=z^6+3z$, $z=0$

$f'(z)=6z^5+3$, so $f'(0)=3=3e^{0i}$ and therefore

f rotates tangent vectors at 0 through an angle of $\theta=0$ and stretches them by a factor of $r=3$.

c) $f(z)=\dfrac{z^2+z+1}{z-1}$, $z=0$

$f'(z)=\dfrac{(z-1)(2z+1)-(z^2+z+1)\cdot 1}{(z-1)^2}$, so $f'(0)=-2=2e^{\pi i}$ and therefore

f rotates tangent vectors at 0 through an angle of $\theta=\pi$ and stretches them by a factor of $r=2$.

(10) $f(z)=|z|$ is not analytic (at any point in $\mathbb{C}$).

PF $f(z)=|z|=\sqrt{x^2+y^2}$, so $f=u+iv$ where $u=\sqrt{x^2+y^2}$ and $v=0$.

Then $u_x=\dfrac{x}{\sqrt{x^2+y^2}}$ and $u_y=\dfrac{y}{\sqrt{x^2+y^2}}$ for $(x,y)\neq(0,0)$, and $v_x=0=v_y$;

so $u_x=v_y$ and $u_y=-v_x \implies x=0$ and $y=0$,

Therefore f is not differentiable at z if $z\neq0$,

so f is not analytic at any $z\in\mathbb{C}$.

REMARK Notice that f is not differentiable at 0,

since $f'(0)=\lim_{z\to0}\dfrac{f(z)-f(0)}{z-0}=\lim_{z\to0}\dfrac{|z|}{z}$ does not exist!

i) for $z\in\mathbb{R}$ with $z>0$, $\dfrac{|z|}{z}=1$ and

ii) for $z\in\mathbb{R}$ with $z<0$, $\dfrac{|z|}{z}=-1$.

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(19) $f(z) = \dfrac{z+1}{z-1}$

a) f is analytic on $\mathbb{C} - \{1\}$ (where $z - 1 \neq 0$)

b) Since $f'(z) = -\dfrac{2}{(z-1)^2}$, $f'(0) = -2 \neq 0$ and hence f is conformal at 0.

c) i) Image of x-axis: if $z = x \in \mathbb{R}$, then $f(z) = \dfrac{x+1}{x-1}$; and for $t \in \mathbb{R}$,

$\dfrac{x+1}{x-1} = t$ iff $x + 1 = t(x-1)$ iff $1 + t = (t-1)x$ iff $x = \dfrac{t+1}{t-1}$ and $t \neq 1$.

Therefore the image of the x-axis is the x-axis with 1 excluded:

$\{z : z = x + yi,\ y = 0 \text{ and } x \neq 1\}$.

ii) Image of y-axis: if $z = yi$, then

$$f(z) = \frac{yi+1}{yi-1} = \frac{1+yi}{-1+yi} \cdot \frac{-1-yi}{-1-yi} = \frac{y^2-1-2yi}{y^2+1} = \frac{y^2-1}{y^2+1} - \frac{2y}{y^2+1} i; \text{ and}$$

$$u = \frac{y^2-1}{y^2+1} \text{ and } v = -\frac{2y}{y^2+1} \text{ gives } u^2+v^2 = \frac{(y^2-1)^2}{(y^2+1)^2} + \frac{(2y)^2}{(y^2+1)^2} = \frac{(y^2+1)^2}{(y^2+1)^2} = 1.$$

Therefore the image lies on the unit circle.

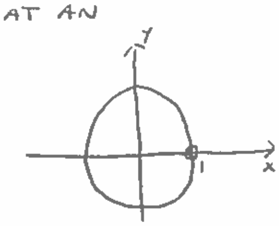
If $(\cos\theta, \sin\theta)$ is any point on the circle other than $(1,0)$,

solving $\cos\theta = \dfrac{y^2-1}{y^2+1}$ gives $y = \pm\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}}$ (with the sign determined by the sign of $\sin\theta$);

so this gives a point $z = yi$ which maps to $(\cos\theta, \sin\theta)$.

Therefore the image of the y-axis is the unit circle with the point $(1,0)$ deleted: $\{z : |z| = 1 \text{ and } z \neq 1\}$.

d) Since f is conformal at 0 and the x-axis and y-axis intersect at an angle of $\frac{\pi}{2}$, their images also intersect at an angle of $\frac{\pi}{2}$.



(20) If f is analytic on a region A and $f^{(n+1)}(z) = 0$ on A, then f is a polynomial of degree at most n.

PF By induction on n:

1) This is true for $n = 0$, since $f'(z) = 0$ on A implies that f is constant on A.

2) Assume this statement is true for an integer n, where $n \geq 0$, and let $f^{(n+2)}(z) = 0$ on A.

If $g(z) = f^{(n+1)}(z)$, then $g'(z) = 0$ on A so $g(z) = C$ on A for some constant C,

If $h(z) = f(z) - \dfrac{Cz^{n+1}}{(n+1)!}$, then $h^{(n+1)}(z) = f^{(n+1)}(z) - C = g(z) - C = 0$ on A;

so by the induction hypothesis $h(z)$ is a polynomial of degree at most n.

Thus $f(z) = h(z) + \dfrac{Cz^{n+1}}{(n+1)!}$ is a polynomial of degree at most $n+1$,

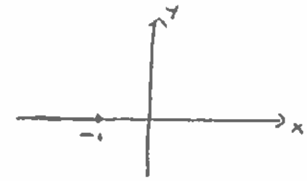
so the assertion is valid for $n+1$.

REMARK Here we are using that $D_z^n\left(\dfrac{z^n}{n!}\right) = 1$ for any $n \in \mathbb{N}$, which follows by induction.

MATH 185A SOLUTIONS — SEC. 1.6 AND CH. 1 REVIEW EX.

1.6 - (2b) $f(z) = \text{LOG}(z+1)$ GIVES $f'(z) = \frac{1}{z+1}$.

IF WE TAKE THE PRINCIPAL BRANCH OF LOG z, THEN f IS ANALYTIC FOR $z+1 \notin L = \{z : x \le 0 \text{ AND } y = 0\}$, SO FOR $z \in \mathbb{C} - \{z : x \le -1 \text{ AND } y = 0\}$.



(3a) $$\lim_{z \to 0} \frac{e^z - 1}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} \quad \text{FOR } f(z) = e^z,$$

SO $$\lim_{z \to 0} \frac{e^z - 1}{z} = f'(0) = e^0 = 1 \quad \text{SINCE } D_z(e^z) = e^z.$$

(4a) $$\lim_{z \to 1} \frac{\text{LOG } z}{z - 1} = \lim_{z \to 1} \frac{f(z) - f(1)}{z - 1} \quad \text{FOR } f(z) = \text{LOG } z,$$

SO $$\lim_{z \to 1} \frac{\text{LOG } z}{z - 1} = f'(1) = \frac{1}{1} = 1 \quad \text{SINCE } D_z(\text{LOG } z) = \frac{1}{z}.$$

CH. 1 RE - (15) LET f BE ANALYTIC ON A, AND DEFINE $g : A \to \mathbb{C}$ BY $g(z) = \overline{f(z)}$. WHEN IS g ANALYTIC?

LET $f = u + iv$, SO $u_x = v_y$ AND $u_y = -v_x$ SINCE f IS ANALYTIC ON A. THEN $g = u - iv = u + iv^*$ WHERE $v^* = -v$,

SO g IS ANALYTIC IFF $u_x = v^*_y$ AND $u_y = -v^*_x$

IFF $u_x = -v_y$ AND $u_y = v_x$

IFF $v_y = -v_y$ AND $-v_x = v_x$ IFF $v_y = 0 = v_x$.

THEN $f' = f_x = u_x + iv_x = 0$, SO f IS CONSTANT ON A (ASSUMING A IS A CONNECTED OPEN SUBSET OF $\mathbb{C}$).

(17) $f(x+iy) = (x^2 + 2y) + i(x^2 + y^2)$. WHERE DOES $f'(z_0)$ EXIST?

SINCE $u_x = 2x$ AND $v_y = 2y$

AND $u_y = 2$ AND $v_x = 2x$,

$u_x = v_y$ AND $u_y = -v_x$ IFF $x = y$ AND $x = -1$,

SO f IS DIFFERENTIABLE ONLY AT $(-1, -1)$.

REMARK NOTICE THAT f IS DIFFERENTIABLE AT $(-1,-1)$, AND SINCE IT HAS CONTINUOUS FIRST PARTIALS AND SATISFIES THE CAUCHY - RIEMANN EQUATIONS THERE.