MATH 185A   SOLUTIONS – SEC. 1.6

(6) IF $\sin z = w$, THEN $\cos^2 z = 1 - \sin^2 z = 1 - w^2$ SO $\cos z = \pm\sqrt{1-w^2}$

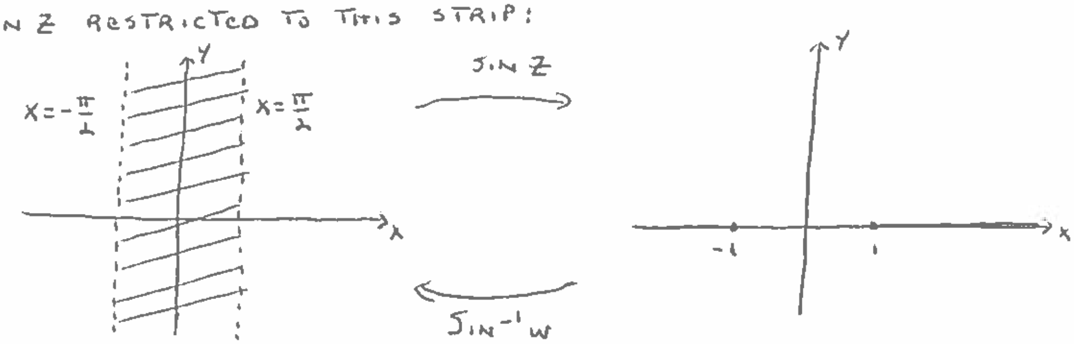
THEN $e^{iz} = \cos z + i \sin z = \pm\sqrt{1-w^2} + iw$,

SO $iz = \operatorname{Log}(iw \pm \sqrt{1-w^2})$ AND $\underline{z = -i \operatorname{Log}(iw \pm \sqrt{1-w^2})}$.

IF WE CHOOSE THE PRINCIPAL BRANCH OF $\operatorname{Log} z$ AND OF $\sqrt{z}$ (AND THE PLUS SIGN), WE GET THE PRINCIPAL BRANCH OF THE INVERSE SINE:

$$\boxed{\sin^{-1} w = -i \operatorname{Log}(iw + \sqrt{1-w^2})}.$$

USING #34 AND #35 IN 1.3, $\sin z$ IS A 1-1 MAP FROM THE STRIP $\{z : -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}\}$ ONTO $\mathbb{C} - \{z : y = 0 \text{ AND } |x| \geq 1\}$, AND $\sin^{-1} w$ IS THE INVERSE OF $\sin z$ RESTRICTED TO THIS STRIP:



USING THE INVERSE FUNCTION THEOREM, WE HAVE

$$\underline{D_w(\sin^{-1} w)} = \frac{1}{D_z(\sin z)} = \frac{1}{\cos z} = \boxed{\frac{1}{\sqrt{1-w^2}}}$$

(USING THE PRINCIPAL BRANCH OF THE SQUARE ROOT)

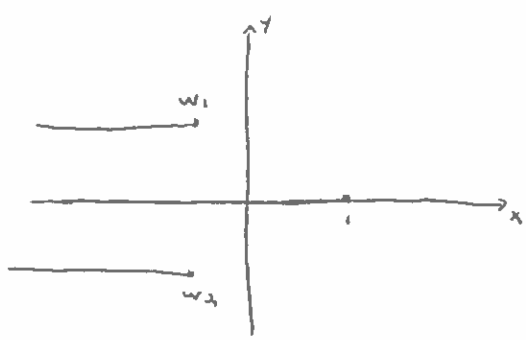
(10a) IF $f(z) = \sqrt{z^3 - 1}$, THEN $\underline{f'(z) = \frac{1}{2}(z^3-1)^{-1/2} \cdot 3z^2}$.

SINCE $z^3 - 1 = (z-1)(z-w_1)(z-w_2)$ WHERE $w_1 = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

AND $w_2 = e^{\frac{4\pi i}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$,

WE HAVE $f(z) = \sqrt{z-1}\,\sqrt{z-w_1}\,\sqrt{z-w_2}$.

USING THE PRINCIPAL BRANCH FOR EACH SQUARE ROOT, WE HAVE THAT f IS ANALYTIC WHERE $z - 1 \notin L$, $z - w_1 \notin L$, AND $z - w_2 \notin L$, WHERE $L = \{x + yi : y = 0 \text{ AND } x \leq 0\}$:



MATH 185A   HOMEWORK SOLUTIONS – SEC. 1.5 AND 2.1

1.5 – 28 $u(x,y) = x^3 - 3xy^2$, so $u_x = 3x^2 - 3y^2$ and $u_y = -6xy$.

Since $u_{xx} + u_{yy} = 6x - 6x = 0$, u is harmonic.

If v is a harmonic conjugate of u, then

$v_y = u_x = 3x^2 - 3y^2$ so $v = 3x^2y - y^3 + g(x)$

Then $v_x = 6xy + g'(x) = -u_y = 6xy$, so $g'(x) = 0$ and $g(x) = k$ for some $k \in \mathbb{R}$.

Thus $v(x,y) = 3x^2y - y^3 + k = \boxed{3x^2y - y^3 + 2}$ since $v(0,0) = 2$.

2.1 – 2b $\int_\gamma (z^2+2z+3)\,dz$ where γ is the line segment from 1 to $2+i$.

$$\int_\gamma (z^2+2z+3)\,dz = \left[\frac{z^3}{3} + z^2 + 3z\right]_1^{2+i} = \frac{1}{3}(2+i)^3 + (2+i)^2 + 3(2+i) - \frac{1}{3} - 1 - 3$$

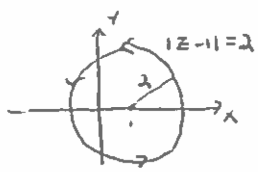
$$= \frac{1}{3}\left(2^3 + 3\cdot 2^2 i + 3\cdot 2\cdot i^2 + i^3\right) + (3+4i) + 6 + 3i - \frac{13}{3}$$

$$= \frac{1}{3}(2+11i) + (3+4i) + 6 + 3i - \frac{13}{3} = \boxed{\frac{16}{3} + \frac{32}{3}i} = \boxed{\frac{16}{3}(1+2i)}$$

(using the Fund. Th. of Calculus for contour integrals)

2c $\int_\gamma \frac{1}{z-1}\,dz$ using $\gamma(\theta) = 1 + 2e^{i\theta}$, $0 \le \theta \le 2\pi$, gives

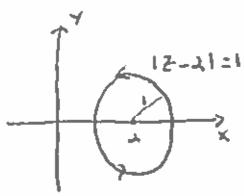
$$\int_\gamma \frac{1}{z-1}\,dz = \int_0^{2\pi} \frac{1}{2e^{i\theta}} \cdot 2ie^{i\theta}\,d\theta = \int_0^{2\pi} i\,d\theta = \boxed{2\pi i}$$



4 $\int_\gamma \frac{1}{z^2-2z}\,dz$ using partial fractions, $\frac{1}{z^2-2z} = \frac{A}{z} + \frac{B}{z-2}$ gives $A = -\frac{1}{2}$, $B = \frac{1}{2}$

so $\int_\gamma \frac{1}{z^2-2z}\,dz = \int_\gamma \left(\frac{-1/2}{z} + \frac{1/2}{z-2}\right)dz$

$= -\frac{1}{2}\int_\gamma \frac{1}{z}\,dz + \frac{1}{2}\int_\gamma \frac{1}{z-2}\,dz$ where



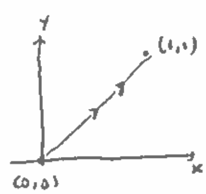
a) $\int_\gamma \frac{1}{z}\,dz = \left[\text{Log } z\right]_3^3 = 0$ since $(\text{Log } z)' = \frac{1}{z}$ on an open set containing γ, and

b) $\gamma(\theta) = 2 + e^{i\theta}$, $0 \le \theta \le 2\pi$, gives

$$\int_\gamma \frac{1}{z-2}\,dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} \cdot ie^{i\theta}\,d\theta = \int_0^{2\pi} i\,d\theta = 2\pi i\ ;$$

Therefore $\int_\gamma \frac{1}{z^2-2z}\,dz = -\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 2\pi i = \boxed{\pi i}$

8 a)



Let $\gamma(t) = t + ti$, $0 \le t \le 1$; then

$$\int_\gamma \bar{z}^2\,dz = \int_0^1 (t - ti)^2 \cdot (1+i)\,dt = \int_0^1 (1-i)^2(1+i)t^2\,dt$$

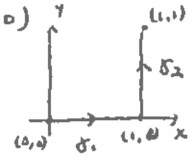
$$= -2i(1+i)\int_0^1 t^2\,dt = (2-2i)\left[\frac{t^3}{3}\right]_0^1 = (2-2i)\cdot\frac{1}{3} = \boxed{\frac{2}{3}(1-i)}$$

$$= \boxed{\frac{2}{3} - \frac{2}{3}i}$$

MATH 185A

SOLUTIONS — 2.1 AND 2.2

2.1 – 8 a)



Let $\gamma_1(t) = t$ for $0 \le t \le 1$ and $\gamma_2(t) = 1 + ti$ for $0 \le t \le 1$; then

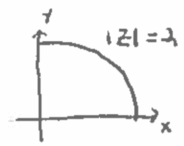
$$\int_\gamma \bar{z}^2\,dz = \int_{\gamma_1} \bar{z}^2\,dz + \int_{\gamma_2} \bar{z}^2\,dz = \int_0^1 t^2\,dt + \int_0^1 (1-ti)^2 \cdot i\,dt$$

$$= \frac{1}{3} + i\int_0^1 (1 - 2ti - t^2)\,dt$$

$$= \frac{1}{3} + i\left[t - it^2 - \frac{t^3}{3}\right]_0^1 = \frac{1}{3} + i\left(\frac{2}{3} - i\right) = \boxed{\frac{4}{3} + \frac{2}{3}i} = \boxed{\frac{2}{3}(2+i)}$$

Since a) and b) have different answers,
$\bar{z}^2$ is NOT the derivative of an analytic function.

10

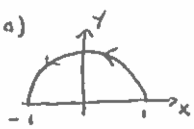


Since $|z^2+1| = |z^2 - (-1)| \ge |z^2| - |-1| = |z|^2 - 1 = 2^2 - 1 = 3$ on C
(using the reverse triangle inequality),

$$\left|\frac{1}{z^2+1}\right| = \frac{1}{|z^2+1|} \le \frac{1}{3} \text{ on } C \text{ and therefore}$$

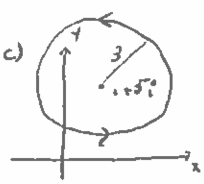
$$\left|\int_C \frac{1}{z^2+1}\,dz\right| \le M\ell = \frac{1}{3}\left(\frac{1}{4}\cdot 2\pi \cdot 2\right) = \frac{\pi}{3} \text{ by the } M\ell\text{-lemma.}$$

2.2 – 1 a)



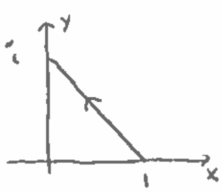
$$\int_\gamma (z^3+3)\,dz = \left[\frac{z^4}{4} + 3z\right]_1^{-1} = \left(\frac{1}{4} - 3\right) - \left(\frac{1}{4} + 3\right) = \underline{-6}$$

1 c)



$\int_\gamma e^{1/z}\,dz = \underline{0}$ by Cauchy's Theorem, since $e^{1/z}$ is analytic everywhere except at 0, and 0 is outside γ.

6

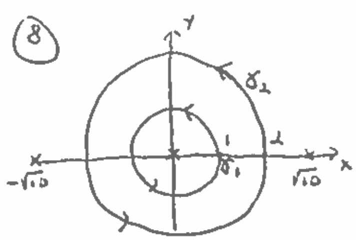


$$\int_\gamma \left(z - \frac{1}{z}\right)dz = \left[\frac{z^2}{2} - \operatorname{Log} z\right]_1^i = \left(-\frac{1}{2} - \frac{\pi}{2}i\right) - \left(\frac{1}{2} - 0\right)$$

$$= \boxed{-1 - \frac{\pi}{2}i}$$

(using the Fund. Th. for contour integrals)

8



$$\int_{\gamma_1} \frac{dz}{z^3(z^2+10)} = \int_{\gamma_2} \frac{dz}{z^3(z^2+10)} \text{ by the Deformation Th.,}$$

since $f(z) = \dfrac{1}{z^3(z^2+10)}$ is analytic on $A = \mathbb{C} - \{0, \pm\sqrt{10}\}$,

γ_1 and γ_2 are in the region A, and

γ_1 can be continuously deformed into γ_2 in A.