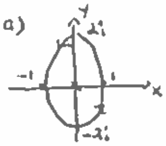
MATH 185A SOLUTIONS — SEC. 2.3 AND 2.4

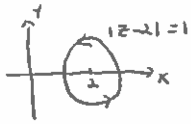
2.3 – 7 a)



$$\int_\gamma \frac{dz}{z} = \underline{2\pi i}$$ SINCE γ CAN BE DEFORMED INTO $|z|=1$ (WITHOUT PASSING THROUGH 0)

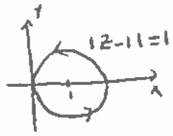
b) $$\int_\gamma \frac{dz}{z^2} = \underline{0}$$ BY THE FUNDAMENTAL TH. OF CONTOUR INTEGRALS SINCE $D_z\left(-\frac{1}{z}\right) = \frac{1}{z^2}$

c)



$$\int_\gamma \frac{e^z}{z}\, dz = \underline{0}$$ BY CAUCHY'S TH. SINCE $\frac{e^z}{z}$ IS ANALYTIC ON AND INSIDE γ

d)

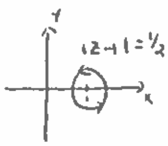


$$\int_\gamma \frac{dz}{z^2-1} = -\frac{1}{2}\int_\gamma \frac{1}{z+1}\, dz + \frac{1}{2}\int_\gamma \frac{1}{z-1}\, dz = -\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 2\pi i = \underline{\pi i}$$

(SINCE THE FIRST INTEGRAL IS 0 BY CAUCHY'S TH, AND

$\frac{1}{z^2-1} = \frac{A}{z-1} + \frac{B}{z+1}$ WHERE $1 = A(z+1) + B(z-1)$ SO $1 = 2A$ AND $1 = -2B$)

10 b)

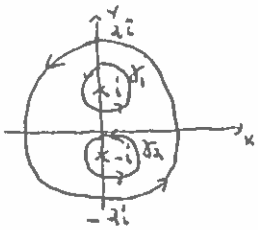


$$\int_\gamma \frac{dz}{(1-z)^3} = \underline{0}$$ BY THE FUNDAMENTAL TH. OF CONTOUR INTEGRALS,

SINCE $F(z) = \frac{1}{2}(1-z)^{-2}$ IS AN ANTIDERIVATIVE FOR $\frac{1}{(1-z)^3}$

ON AN OPEN SET CONTAINING γ.

2.4 – 2 a)



IF γ_1 AND γ_2 ARE SMALL CIRCLES CENTERED AT i AND $-i$, RESPECTIVELY,

$$\int_\gamma \frac{z^2-1}{z^2+1}\, dz = \int_{\gamma_1} \frac{z^2-1}{z^2+1}\, dz + \int_{\gamma_2} \frac{z^2-1}{z^2+1}\, dz$$

$$= \int_{\gamma_1} \frac{(z^2-1)/(z+i)}{z-i}\, dz + \int_{\gamma_2} \frac{(z^2-1)/(z-i)}{z+i}\, dz$$

$$= 2\pi i\left(\frac{z^2-1}{z+i}\Big|_{z=i}\right) + 2\pi i\left(\frac{z^2-1}{z-i}\Big|_{z=-i}\right) = -2\pi + 2\pi = \underline{0}$$

OR $$\int_\gamma \frac{z^2-1}{z^2+1}\, dz = \int_\gamma \left(1 - \frac{2}{z^2+1}\right) dz$$ WHERE $\frac{2}{z^2+1} = \frac{A}{z-i} + \frac{B}{z+i}$

$2 = A(z+i) + B(z-i)$

$z=i$: $2 = 2iA$ $\quad \underline{A = -i}$

$z=-i$: $2 = -2iB$ $\quad \underline{B = i}$

$$\int_\gamma \frac{z^2-1}{z^2+1}\, dz = \int_\gamma \left(1 + \frac{i}{z-i} - \frac{i}{z+i}\right) dz$$

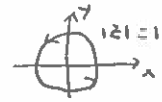
$$= \int_\gamma 1\, dz + i\int_\gamma \frac{1}{z-i}\, dz - i\int_\gamma \frac{1}{z+i}\, dz$$

$$= 0 + i(2\pi i) - i(2\pi i) = \underline{0}$$

(WHERE THE FIRST TERM IS 0 BY THE FUNDAMENTAL TH FOR CONTOUR INTEGRALS.)

MATH 185A SOLUTIONS – SEC. 2.4

(2) b) $\displaystyle\int_\gamma \frac{\sin e^z}{z}\,dz = 2\pi i\left(\sin e^z\Big|_{z=0}\right) = \boxed{2\pi i \sin 1}$



(3) f is entire with $|f(z)| \le M|z|^n$ for large z, for some M and some integer $n \ge 0$. Then f is a polynomial of degree at most n.

PF Fix z_0, and consider the circle $|z - z_0| = R$ for R large.

Since $|z| = |z - z_0 + z_0| \le |z - z_0| + |z_0| = R + |z_0|$ on the circle,

Cauchy's inequality gives

$$|f^{(n+1)}(z_0)| \le \frac{(n+1)!}{R^{n+1}}\left(M(R+|z_0|)^n\right) = \frac{(n+1)!}{R}\cdot\frac{(R+|z_0|)^n}{R^n}$$

$$= \frac{(n+1)!}{R}\cdot\left(1+\frac{|z_0|}{R}\right)^n \longrightarrow 0\cdot 1 = 0 \quad \text{as } R\to\infty,$$

Therefore $f^{(n+1)}(z_0) = 0$ for any z_0,

so f is a polynomial with $\deg(f) \le n$ by #20 in Sec. 1.5.

OR PF (by induction on n)

1) If $n = 0$, $|f(z)| \le M$ for $|z| > r$ for some r.

Since f is continuous and $\{z : |z| \le r\}$ is compact,

$|f(z)| \le K$ for $|z| \le r$ for some constant K.

Therefore f is bounded and entire,

so f is constant by Liouville's Th.

2) Suppose the assertion holds for $n-1$, and

let $|f(z)| \le M|z|^n$ for large z.

Let $g(z) = \begin{cases} \dfrac{f(z)-f(0)}{z}, & \text{if } z \ne 0 \\ f'(0), & \text{if } z = 0. \end{cases}$

Then g is analytic on $\mathbb{C} - \{0\}$ and is continuous at 0,

so g is analytic on $\mathbb{C}$ with

$$|g(z)| = \left|\frac{f(z)}{z} - \frac{f(0)}{z}\right| \le \left|\frac{f(z)}{z}\right| + \left|\frac{f(0)}{z}\right| = \frac{|f(z)|}{|z|} + \frac{|f(0)|}{|z|}$$

$$\le \frac{M|z|^n}{|z|} + \frac{|f(0)||z|^n}{|z|} = (M + |f(0)|)\,|z|^{n-1} \quad \text{for } z \text{ large.}$$

Therefore g is a polynomial with $\deg(g) \le n-1$, so

$f(z) = z\,g(z) + f(0)$ is a polynomial with $\deg(f) \le n$.

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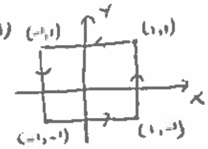
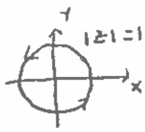
(5) a)



$\int_\gamma \frac{dz}{z^3} = \underline{0}$ BY THE FUND. TH. OF CONTOUR INTEGRALS SINCE $F(z) = -\frac{1}{2z^2}$ IS AN ANTIDERIVATIVE FOR $\frac{1}{z^3}$ ON AN OPEN SET CONTAINING γ.

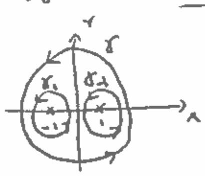
(OR USE $\int_\gamma \frac{dz}{z^3} = \frac{2\pi i}{2!}\left(D_z^2(1)\right) = \underline{0}$)

b)



$$\int_\gamma \frac{\sin z}{z^4}\, dz = \frac{2\pi i}{3!}\left(D_z^3(\sin z)\Big|_{z=0}\right) = \frac{2\pi i}{6}\left((-\cos z)\Big|_{z=0}\right) = \boxed{-\frac{\pi i}{3}}$$

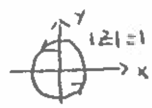
(13) a) $$\int_\gamma \frac{dz}{z^2-1} = -\frac{1}{2}\int_\gamma \frac{1}{z-1}\,dz + \frac{1}{2}\int_\gamma \frac{1}{z+1}\,dz = -\frac{1}{2}(2\pi i) + \frac{1}{2}(2\pi i) = \underline{0}$$



OR $$\int_\gamma \frac{1}{z^2-1}\,dz = \int_{\gamma_1} \frac{dz}{z^2-1} + \int_{\gamma_2} \frac{dz}{z^2-1}$$
$$= \int_{\gamma_1} \frac{\frac{1}{z-1}}{z+1}\,dz + \int_{\gamma_2} \frac{\frac{1}{z+1}}{z-1}\,dz$$
$$= 2\pi i\left(\frac{1}{z-1}\Big|_{z=-1}\right) + 2\pi i\left(\frac{1}{z+1}\Big|_{z=1}\right) = -\pi i + \pi i = \underline{0}$$

d) $$\int_\gamma \frac{dz}{z^2+2z-3} = \int_\gamma \frac{dz}{(z+3)(z-1)} = \int_\gamma \frac{\frac{1}{z+3}}{z-1}\,dz = 2\pi i\left(\frac{1}{z+3}\Big|_{z=1}\right) = \frac{2\pi i}{4} = \boxed{\frac{\pi i}{2}}$$

(14) $\int_\gamma \frac{e^z}{z}\,dz = 2\pi i\left(e^z\big|_{z=0}\right) = \underline{2\pi i}$, SO



$$\int_0^{2\pi} \frac{e^{e^{i\theta}}}{e^{i\theta}}\cdot ie^{i\theta}\,d\theta = i\int_0^{2\pi} e^{\cos\theta + i\sin\theta}\,d\theta = i\int_0^{2\pi} e^{\cos\theta}\left(\cos(\sin\theta) + i\sin(\sin\theta)\right)d\theta = 2\pi i.$$

THEN $-\int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta)\,d\theta + i\int_0^{2\pi} e^{\cos\theta}(\cos(\sin\theta))\,d\theta = 2\pi i$, SO

$\underline{\int_0^{2\pi} e^{\cos\theta}(\cos(\sin\theta))\,d\theta = 2\pi}$ (AND $\int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta)\,d\theta = 0$).

LETTING $\underline{u = 2\pi - \theta}$ GIVES $\int_\pi^{2\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta = \int_0^\pi e^{\cos u}\cos(-\sin u)\,du$
$= \int_0^\pi e^{\cos u}\cos(\sin u)\,du$,

SO $2\int_0^\pi e^{\cos\theta}\cos(\sin\theta)\,d\theta = 2\pi$ AND $\int_0^\pi e^{\cos\theta}\cos(\sin\theta)\,d\theta = \underline{\pi}$.

OR USE $\gamma(\theta) = e^{i\theta}$ FOR $-\pi \le \theta \le \pi$ TO PARAMETRIZE THE CIRCLE,

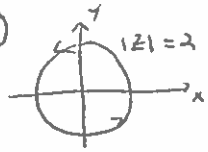
SO $\underline{\int_{-\pi}^{\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta = 2\pi}$ GIVES

$2\int_0^\pi e^{\cos\theta}\cos(\sin\theta)\,d\theta = 2\pi$ (SINCE $e^{\cos\theta}\cos(\sin\theta)$ IS AN EVEN FUNCTION),

SO $\int_0^\pi e^{\cos\theta}\cos(\sin\theta)\,d\theta = \underline{\pi}.$

MATH 185A $\qquad$ SOLUTIONS – Sec. 2.4

(15)



$$\int_C \frac{|z|e^z}{z^2}\,dz = \int_C \frac{2e^z}{z^2}\,dz = \frac{2\pi i}{1!}\left(D_z(2e^z)\Big|_{z=0}\right) = 2\pi i\left(2e^z\Big|_{z=0}\right) = \underline{4\pi i}$$

(18) a)

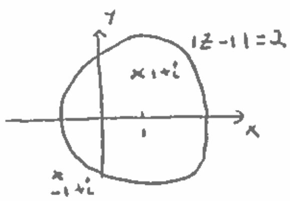
$$\int_{|z|=1} \frac{e^z}{z^2}\,dz = \frac{2\pi i}{1!}\left(D_z(e^z)\Big|_{z=0}\right) = 2\pi i\left(e^z\Big|_{z=0}\right) = \underline{2\pi i}$$

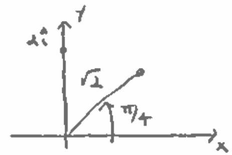
b)

$$\int_{|z|=1} \frac{\cos z}{z^2}\,dz = \frac{2\pi i}{1!}\left(D_z(\cos z)\Big|_{z=0}\right) = 2\pi i\left(-\sin z\Big|_{z=0}\right) = \underline{0}$$

(19a)

$$\int_\gamma \frac{dz}{z^2-2i}$$





$$\sqrt{2i} = \pm\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \pm\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}\right) = \pm(1+i)$$

$$\int_\gamma \frac{dz}{z^2-2i} = \int_\gamma \frac{1}{(z-(1+i))(z+(1+i))}\,dz$$

$$= \int_\gamma \frac{\frac{1}{z+(1+i)}}{z-(1+i)}\,dz = 2\pi i\left(\frac{1}{z+(1+i)}\Bigg|_{z=1+i}\right)$$

$$= 2\pi i\left(\frac{1}{2+2i}\right) = \frac{2\pi i}{2(1+i)} = \frac{\pi i(1-i)}{(1+i)(1-i)} = \boxed{\frac{\pi}{2}(1+i)}$$

(OR use $(x+yi)^2 = 2i$, $\quad x^2-y^2+2xyi = 2i$,

so $x^2-y^2=0$ and $2xy=2$,

$y = \pm x$ and $xy = 1$

if $y = x$, $\quad x^2 = 1$ so $x = \pm 1$ and $\underline{x=y=1}$ or $\underline{x=y=-1}$.

if $y=-x$, $\quad -x^2 = 1$ (no solution)

therefore $x+yi = \underline{1+i}$ or $x+yi = \underline{-1-i}$)