MATH 185A SOLUTIONS – SEC. 2.5

(2) $\underline{\cos z} = \cos(x+iy) = \cos x \cos(iy) - \sin x(\sin iy)$

$= \underline{\cos x \cosh y - i \sin x \sinh y}$, so

$\underline{|\cos z|^2} = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$

$= \cos^2 x \cosh^2 y + \sin^2 x (\cosh^2 y - 1)$

$= (\cos^2 x + \sin^2 x)\cosh^2 y - \sin^2 x = \underline{\cosh^2 y - \sin^2 x}$

Since $\cosh y$ is increasing on $[0, 2\pi]$,

$|\cos z|^2$ has a max. value of $\cosh^2 2\pi$ when $\sin x = 0$; so

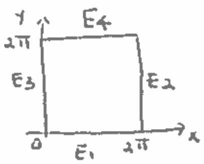
$|\cos z|$ has a max. value of $\boxed{\cosh 2\pi}$ $\underline{\text{at } (0,2\pi),\ (\pi,2\pi), \text{ and } (2\pi,2\pi)}$.

OR $\cos z = \cos x \cosh y - i \sin x \sinh y$, so

$\underline{|\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$.

By the max. modulus principle, $|\cos z|$ attains its max. on $[0,2\pi] \times [0,2\pi]$

at a point on the boundary:



on E_1, $|\cos z|^2 = \cos^2 x$ has a max. value of $\underline{1}$.

on E_2, $|\cos z|^2 = \cosh^2 y$ has a max. value of $\underline{\cosh^2 2\pi}$ at $(2\pi, 2\pi)$.

on E_3, $|\cos z|^2 = \cosh^2 y$ has a max. value of $\underline{\cosh^2 2\pi}$ at $(0, 2\pi)$.

on E_4, $|\cos z|^2 = \cos^2 x \cosh^2 2\pi + \sin^2 x \sinh^2 2\pi$

$= \cos^2 x \cosh^2 2\pi + \sin^2 x(\cosh^2 2\pi - 1) = \cosh^2 2\pi - \sin^2 x$

has a max. value of $\underline{\cosh^2 2\pi}$ at $(0,2\pi)$, $(\pi, 2\pi)$, $(2\pi, 2\pi)$.

Therefore $|\cos z|$ has a max. value of $\boxed{\cosh 2\pi}$ $\underline{\text{at } (0,2\pi),\ (\pi,2\pi), \text{ and } (2\pi,2\pi)}$.

(4a) $T(z) = \dfrac{R(z - z_0)}{R^2 - \bar{z}_0 z}$ Let $\underline{D} = D(0;R)$ and $\underline{D_1} = D(0;1)$.

1) $\underline{T(z_0)} = \dfrac{R(0)}{R^2 - |z_0|^2} = \underline{0}$; and $\underline{T \text{ is analytic on } D}$ since it is a quotient

of 2 analytic functions and $R^2 - \bar{z}_0 z = 0$ iff $z = \dfrac{R^2}{\bar{z}_0}$, and $\dfrac{R^2}{\bar{z}_0} \notin D$

since $\left|\dfrac{R^2}{\bar{z}_0}\right| = \dfrac{R^2}{|\bar{z}_0|} = \dfrac{R^2}{|z_0|} = R\left(\dfrac{R}{|z_0|}\right) > R \cdot 1 = R$ since $|z_0| < R$.

2) $\underline{\text{If } |z| = R, \text{ then } |T(z)| = 1}$ since

$|T(z)|^2 = T(z)\overline{T(z)} = \dfrac{R(z-z_0)}{R^2 - \bar{z}_0 z} \cdot \dfrac{R(\bar{z} - \bar{z}_0)}{R^2 - z_0 \bar{z}} = \dfrac{R^2(R^2 - z_0\bar{z} - \bar{z}_0 z + |z_0|^2)}{R^4 - R^2 z_0 \bar{z} - R^2 \bar{z}_0 z + R^2|z_0|^2} = 1$

(using that $z_0 \bar{z}_0 = |z_0|^2$ and $z\bar{z} = |z|^2 = R^2$).

3) By the max. modulus principle, since T is analytic on D and has a max. modulus of 1 on the boundary, T maps the closed disc $\bar{D} = \{z : |z| \le R\}$ into the closed disc $\bar{D}_1 = \{z : |z| \le 1\}$.

Since T is not constant, it does not attain its max. modulus in the open disc D; so T maps D into D_1.

(CONTINUED ON NEXT PAGE)

4a) 4) Since $T(z) = w$ iff $\dfrac{R(z - z_0)}{R^2 - \bar{z}_0 z} = w$ iff $Rz - Rz_0 = R^2 w - \bar{z}_0 z w$ iff

$Rz + \bar{z}_0 z w = R^2 w + R z_0$ iff $(R + \bar{z}_0 w) z = R(Rw + z_0)$ iff $z = \dfrac{R(Rw + z_0)}{R + \bar{z}_0 w}$

(for $z \neq \dfrac{R^2}{\bar{z}_0}$ and $w \neq -\dfrac{R}{\bar{z}_0}$), $S(w) = \dfrac{R(Rw + z_0)}{R + \bar{z}_0 w}$ is the inverse of T.

5) If $|w| = 1$, then $|S(w)| = R$ since

$$|S(w)|^2 = S(w)\overline{S(w)} = \frac{R(Rw + z_0)}{R + \bar{z}_0 w} \cdot \frac{R(R\bar{w} + \bar{z}_0)}{R + z_0 \bar{w}} = \frac{R^2(R^2 + Rz_0\bar{w} + R\bar{z}_0 w + |z_0|^2)}{R^2 + Rz_0\bar{w} + R\bar{z}_0 w + |z_0|^2} = R^2.$$

Therefore S maps $\bar{D}_1$ into $\bar{D}$ and D_1 into D (since S is not constant) by the max. modulus principle,

so T is a 1-1 onto mapping from D to D_1.

REMARK Notice that T is a fractional linear transformation.
(An alternate way to get this result is to apply Prop. 5.2.2 to $T \circ U$, where $U: \bar{D}_1 \to \bar{D}$ is given by $U(z) = Rz$.)

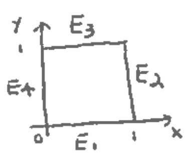
7 Since $f(z) = e^{z^2}$ is analytic on the disc $D(0;1)$, it has its maximum modulus on $\bar{D} = \{z : |z| \leq 1\}$ at a point on the boundary $|z| = 1$.

If $|z| = 1$ with $z = e^{i\theta}$, then $z^2 = e^{2i\theta}$ so $|e^{z^2}| = e^{\operatorname{Re}(z^2)} = e^{\cos 2\theta}$;

so $|e^{z^2}|$ has a max. value of $\boxed{e}$ when $\theta = 0, \pi$, or 2π so $z = \pm 1$.

8 $u = \operatorname{Re} z^3$ on $[0,1] \times [0,1]$

Since u is harmonic, its maximum occurs on the boundary.



Since z^3 maps $re^{i\theta}$ to $r^3 e^{3i\theta}$, $u(z) \leq 0$ on E_3 and E_4;

so we only need to consider u on E_1 and E_2,

where $u = \operatorname{Re}(x + iy)^3 = x^3 - 3xy^2$;

on E_1, $u = x^3$ has a max. of 1 when $x = 1$.

on E_2, $u = 1 - 3y^2$ has a max. of 1 when $y = 0$.

Therefore u has a max. value of $\boxed{1}$ at the point $(1,0)$.

9b $u(x,y) = \operatorname{Log}\sqrt{x^2 + y^2} = \ln r$

Since u is harmonic on $\mathbb{C} - \{0\}$, it has a harmonic conjugate on any simply connected region in $\mathbb{C} - \{0\}$.

Since $u = \ln r$, the Cauchy–Riemann equations in polar form give

$u_r = \frac{1}{r} v_\theta = \frac{1}{r}$ and $v_r = -\frac{1}{r} u_\theta = 0$, so $v_\theta = 1$ and $v_r = 0$.

Therefore $v = \theta + k$ for some $k \in \mathbb{R}$.

To make v continuous, we need to delete a ray with the origin as endpoint: if we delete the ray $L = \{x + iy : x \leq 0 \text{ and } y = 0\}$ and take $-\pi < \theta < \pi$, we get $v = \operatorname{Arg}(z)$ as an example (taking $k = 0$).

In this case, we have
$$v(x,y) = \begin{cases} \tan^{-1}\dfrac{y}{x}, & \text{if } x > 0 \\ \dfrac{\pi}{2} - \tan^{-1}\dfrac{x}{y}, & \text{if } x \leq 0,\ y > 0 \\ -\dfrac{\pi}{2} - \tan^{-1}\dfrac{x}{y}, & \text{if } x \leq 0,\ y < 0, \end{cases} \quad \text{on } \mathbb{C} - L.$$

MATH 185A SOLUTIONS — SEC. 2.5

(12) a) LET u BE THE CONSTANT FUNCTION $u(z)=1$ ON THE CLOSED DISC $\{z : |z| \le R\}$.

BY POISSON'S FORMULA,

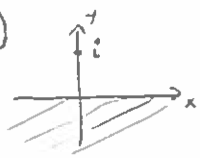
$$1 = u(re^{i\phi}) = \frac{R^2-r^2}{2\pi}\int_0^{2\pi} \frac{1}{R^2-2rR\cos(\phi-\theta)+r^2}\,d\theta \qquad (\text{since } u(Re^{i\theta})=1),$$

$$\text{so} \quad \int_0^{2\pi} \frac{R^2-r^2}{R^2-2rR\cos(\phi-\theta)+r^2}\,d\theta = 2\pi. \qquad (\text{assuming } r<R).$$

b) LET $u_0(z)=x$ ON THE CIRCLE $|z|=1$. SINCE THE FUNCTION $u(z)=x$ IS HARMONIC ON THE CLOSED DISC $\{z : |z| \le 1\}$, IT IS THE UNIQUE SOLUTION TO THE DIRICHLET PROBLEM IN THIS CASE; SO

$$u(re^{i\phi}) = \frac{1^2-r^2}{2\pi}\int_0^{2\pi} \frac{u(1\cdot e^{i\theta})}{1^2-2\cdot 1\cdot r\cos(\phi-\theta)+r^2}\,d\theta \quad \text{GIVES}$$

$$r\cos\phi = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)\cos\theta}{1-2r\cos(\phi-\theta)+r^2}\,d\theta .$$

(18) 

SINCE $\operatorname{Im} f \le 0$, $|f(z)-i| \ge 1$ FOR ALL z AND THEREFORE $\frac{1}{|f(z)-i|} \le 1$ FOR ALL z. SINCE f IS ENTIRE, $\frac{1}{f(z)-i}$ IS ENTIRE ALSO (SINCE $f(z)-i \ne 0$); SO $\frac{1}{f(z)-i}$ IS CONSTANT BY LIOUVILLE'S TH. THEREFORE f IS ALSO CONSTANT.

$\boxed{\text{OR}}$ SINCE f IS ENTIRE, $-if$ AND e^{-if} ARE ENTIRE.

SINCE $|e^{-if}| = e^{\operatorname{Re}(-if)} = e^{\operatorname{Im}(f)} \le e^0 = 1$,

e^{-if} IS CONSTANT BY LIOUVILLE'S TH.

IF $e^{-if} = c$, THEN $-if(z)$ IS A VALUE OF $\log c$ FOR EACH VALUE OF z; AND SINCE $-if$ IS CONTINUOUS, IT MUST BE THE SAME VALUE OF $\log c$ FOR ALL z. THEREFORE f IS CONSTANT.

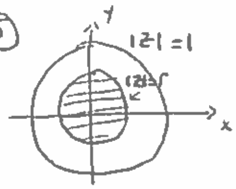
MATH 185A SOLUTIONS – SEC. 3.1

1 a) $z_n = (-1)^n + \frac{i}{n+1}$ DIVERGES SINCE $\mathrm{Re}(z_n) = (-1)^n$ DIVERGES.

b) $z_n = \frac{n!}{n^n} i^n$, SO $|z_n| = \frac{n!}{n^n}$. THEN

$$\frac{|z_{n+1}|}{|z_n|} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n+1}{(n+1)^{n+1}} \cdot n^n = \frac{n^n}{(n+1)^n} = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \longrightarrow \frac{1}{e} < 1,$$

SO $|z_n| \to 0$ AND THEREFORE $z_n \to \boxed{0}$.

5 

a) IF $|z| \le r$, THEN $|z^n - 0| = |z^n| = |z|^n \le r^n$ WHERE $r^n \to 0$ SINCE $0 < r < 1$, SO $f_n \to 0$ UNIFORMLY ON D_r (WHERE $f_n(z) = z^n$).

b) IF $f_n \to 0$ UNIFORMLY ON D, THEN THERE IS AN INTEGER N SUCH THAT $n \ge N \Rightarrow |z^n - 0| = |z|^n < \frac{1}{2}$ FOR ALL z IN D. LET $T = \frac{1}{2N}$ AND LET $z = 1 - T$; THEN $(1-T)^N \ge 1 - NT = \frac{1}{2}$ (BY BERNOULLI'S INEQUALITY), AND THIS GIVES A CONTRADICTION. THEREFORE THE CONVERGENCE IS NOT UNIFORM ON D.

7 a) $\sum_{n=2}^{\infty} \frac{i^n}{\log n}$

1) THE SERIES DOES NOT CONVERGE ABSOLUTELY, SINCE $\sum_{n=2}^{\infty} |z_n| = \sum_{n=2}^{\infty} \frac{|i^n|}{\log n} = \sum_{n=2}^{\infty} \frac{1}{\log n}$ DIVERGES BY THE COMPARISON TEST: $\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES (HARMONIC SERIES), AND $\frac{1}{\log n} \ge \frac{1}{n}$ FOR $n \ge 2$.

2) $\sum_{n=2}^{\infty} \frac{i^n}{\log n}$ CONVERGES SINCE

$$\sum_{n=2}^{\infty} \frac{i^n}{\log n} = -\frac{1}{\log 2} - \frac{i}{\log 3} + \frac{1}{\log 4} + \frac{i}{\log 5} - \frac{1}{\log 6} - \frac{i}{\log 7} + \frac{1}{\log 8} - \cdots$$

$$= \left(-\frac{1}{\log 2} + \frac{1}{\log 4} - \frac{1}{\log 6} + \cdots\right) + i\left(-\frac{1}{\log 3} + \frac{1}{\log 5} - \frac{1}{\log 7} + \cdots\right)$$

WHERE $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\log(2n-2)} = -\frac{1}{\log 2} + \frac{1}{\log 4} - \frac{1}{\log 6} + \cdots$

AND $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\log(2n-1)} = -\frac{1}{\log 3} + \frac{1}{\log 5} - \frac{1}{\log 7} + \cdots$

BOTH CONVERGE BY THE ALTERNATING SERIES TEST.

7 b) $\displaystyle\sum_{n=1}^{\infty} \frac{i^n}{n}$

1) Since $\displaystyle\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \frac{|i^n|}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges (the harmonic series),

the series does not converge absolutely.

2) $\displaystyle\sum_{n=1}^{\infty} \frac{i^n}{n} = \frac{i}{1} - \frac{1}{2} - \frac{i}{3} + \frac{1}{4} + \frac{i}{5} - \frac{1}{6} - \frac{i}{7} + \frac{1}{8} - \cdots$

$\displaystyle = \left(-\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \cdots\right) + i\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \cdots\right)$ converges,

since $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{2n} = -\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \cdots$

and $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n-1} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \cdots$

both converge by the Alternating Series Test.

10 a) Since $\displaystyle\sum_{k=1}^{\infty} a_k$ converges, given $\varepsilon > 0$ there is an N such that

$\displaystyle m > n \geq N \Rightarrow \left|\sum_{k=n+1}^{m} a_k\right| < \varepsilon$. Therefore $n \geq N \Rightarrow |a_{n+1}| < \varepsilon$ (taking $m = n+1$),

so $a_n \to 0$.

OR Let $a_k = x_k + i y_k$,

so $\displaystyle\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} x_k + i \sum_{k=1}^{\infty} y_k$. Then $\displaystyle\sum_{k=1}^{\infty} x_k$ and $\displaystyle\sum_{k=1}^{\infty} y_k$ both converge,

so $x_k \to 0$ and $y_k \to 0$ and therefore $a_k \to 0$.

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b) Assume that $\displaystyle\sum_{k=1}^{\infty} g_k(z)$ converges uniformly on a set A.

Then it is uniformly Cauchy on A, so given $\varepsilon > 0$ there is

an N such that $\displaystyle m > n \geq N \Rightarrow \left|\sum_{k=n+1}^{m} g_k(z)\right| < \varepsilon$ for all z in A.

Therefore $n \geq N \Rightarrow |g_{n+1}(z)| < \varepsilon$ for all z in A (taking $m = n+1$),

so $g_n(z) \to 0$ uniformly on A.