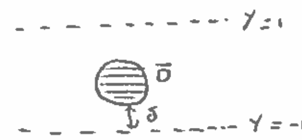
MATH 185A SOLUTIONS – Sec. 3.1

(13) Let $A = \{z : -1 < \operatorname{Im} z < 1\}$, and let $\overline{D}$ be a closed disc in A.



By the Distance Lemma there is a $\delta > 0$ such that every point in $\overline{D}$ is at least a distance δ from A^c;

so $\underline{-1+\delta \le y \le 1-\delta}$ for all $z \in \overline{D}$ with $z = x+yi$.

Then $ny \le n(1-\delta)$ and $-ny \le -n(-1+\delta) = n(1-\delta)$, so

$$|\sin nz| = \left|\frac{e^{inz}-e^{-inz}}{2i}\right| \le \frac{|e^{inz}|+|e^{-inz}|}{2} = \frac{e^{-ny}+e^{ny}}{2} \le \frac{2e^{n(1-\delta)}}{2} = e^{n(1-\delta)} \text{ if } z\in\overline{D},$$

so 1) $|e^{-n}\sin nz| = e^{-n}|\sin nz| \le e^{-n}e^{n(1-\delta)} = e^{-n\delta}$ for all $z \in \overline{D}$

and 2) $\sum_{n=1}^{\infty} e^{-n\delta} = \sum_{n=1}^{\infty}(e^{-\delta})^n$ converges since it is a geometric series with $|r| = e^{-\delta} < e^0 = 1$.

Therefore $\sum_{n=1}^{\infty} e^{-n}\sin nz$ converges uniformly on $\overline{D}$ by the Weierstrass M-Test, so

$\sum_{n=1}^{\infty} e^{-n}\sin nz$ is analytic on A by the Analytic Convergence Th. since each term $g_n = e^{-n}\sin nz$ is analytic on $\mathbb{C}$.

(17) Let $f_n(z) = \sum_{k=1}^{n} \frac{z^k}{k^2}$, and let $A = D(0;1)$.

Then $f_n \to f$ uniformly on A where $f(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^2}$

by the Weierstrass M-Test since

1) $\left|\frac{z^k}{k^2}\right| = \frac{|z|^k}{k^2} \le \frac{1}{k^2}$ for all z in A and

2) $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges (p-series, $p>1$),

and each f_n is analytic on A (since it is a polynomial).

$\underline{\text{However, } f_n' \text{ does not converge to } f' \text{ uniformly on } A:}$

$f_n'(z) = \sum_{k=1}^{n}\frac{z^{k-1}}{k}$, and if (f_n') converged uniformly it would be uniformly Cauchy. Therefore there would be an integer N such that $\left|\sum_{k=n+1}^{m}\frac{z^{k-1}}{k}\right| < 1$ for $m > n \ge N$ and all $z \in A$ (taking $\varepsilon = 1$).

Then for $z = x \in (0,1)$, we have $\underline{\frac{x^n}{n+1} + \frac{x^{n+1}}{n+2} + \cdots + \frac{x^{m-1}}{m} < 1 \text{ for } m > n \ge N.}$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, we can find an integer p such that $\frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{N+p+1} > 2$.

If we choose $x \in (0,1)$ with $x^{N+p} > \frac{1}{2}$, we have that

$$\frac{x^N}{N+1} + \frac{x^{N+1}}{N+2} + \cdots + \frac{x^{N+p}}{N+p+1} > x^{N+p}\left(\frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{N+p+1}\right) > \frac{1}{2}(2) = 1;$$

and this gives a contradiction.

MATH 185A SOLUTIONS – Sec. 3.2

(6) $1 + e^z = 1 + \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) = 2 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots,$

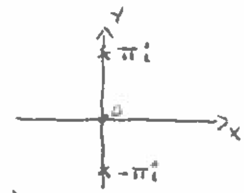
so dividing gives

$$\begin{array}{r|l} & \frac{1}{2} - \frac{1}{4}z + \frac{1}{48}z^3 + \cdots \\ \hline 2 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots & 1 \\ & 1 + \frac{1}{2}z + \frac{1}{4}z^2 + \frac{1}{12}z^3 + \cdots \\ \hline & -\frac{1}{2}z - \frac{1}{4}z^2 - \frac{1}{12}z^3 - \cdots \\ & -\frac{1}{2}z - \frac{1}{4}z^2 - \frac{1}{8}z^3 - \cdots \\ \hline & \frac{1}{24}z^3 + \cdots \end{array}$$

so $\frac{1}{1+e^z} = \boxed{\frac{1}{2} - \frac{1}{4}z + \frac{1}{48}z^3 + \cdots}$

Since $e^z = -1$ iff $z = \mathrm{Log}(-1) = \pi i + 2n\pi i = (2n+1)\pi i$ for $n \in \mathbb{Z}$,

the radius $\boxed{R = \pi}$, since this is the distance from $z_0 = 0$ to the closest singularity.



(or use Taylor's formula to find the first 4 terms of the series.)

(7) b) $\frac{1}{(z-1)(z-2)}, \quad z_0 = 0$

$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$ $\qquad 1 = A(z-2) + B(z-1)$

$z=2: \quad 1 = B$

$z=1: \quad 1 = -A \quad A = -1$

so $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

$= -\frac{1}{2-z} + \frac{1}{1-z}$ $\qquad$ (for $|z|<1$) $\qquad$ (for $|z|<2$)

$= \frac{1}{1-z} - \frac{1/2}{1-z/2} = \sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$

$= \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$

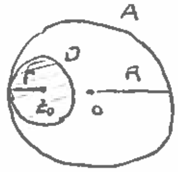
$= \boxed{\sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right)z^n}$ for $|z| < 1$

(8a) $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots,$ so

$\sin z^2 = \sum_{n=0}^{\infty} (-1)^n \frac{(z^2)^{2n+1}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{(2n+1)!}}$

$= z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \frac{z^{14}}{7!} + \cdots$

(14) a)



Let $r = R - |z_0|$, and let $D = D(z_0; r)$.

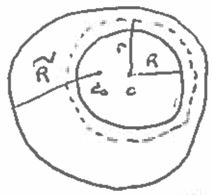
Then $D \subseteq A$ since $z \in D \Rightarrow |z - z_0| < r \Rightarrow$

$$|z| = |z - z_0 + z_0| \le |z - z_0| + |z_0| < r + |z_0| = R \Rightarrow z \in A;$$

so f is analytic on D since f is analytic on A.

Therefore the series for f around z_0 converges on D, so $\underline{\tilde{R} \ge r = R - |z_0|}$.

b)



Suppose that $\tilde{R} > R + |z_0|$, and choose r with $\tilde{R} - |z_0| > r > R$.

If $z \in D(0; r)$, then $z \in D(z_0; \tilde{R})$

since $|z| < r \Rightarrow |z - z_0| \le |z| + |z_0| < r + |z_0| < \tilde{R}$.

Since f is analytic on $D(z_0; \tilde{R})$, f is analytic on $D(0; r)$;

so $R \ge r > R$, which gives a contradiction.

Therefore $\underline{\tilde{R} \le R + |z_0|}$.

(18) Since $\tan z = \dfrac{\sin z}{\cos z}$, dividing gives

$$\begin{array}{r|llll}
 & z & +\frac{z^3}{3} & +\frac{2}{15}z^5 & + \cdots \\
\hline
1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots & z & -\frac{z^3}{6} & +\frac{z^5}{120} & - \cdots \\
 & z & -\frac{z^3}{2} & +\frac{z^5}{24} & - \cdots \\
\hline
 & & \frac{z^3}{3} & -\frac{z^5}{30} & + \cdots \\
 & & \frac{z^3}{3} & -\frac{z^5}{6} & + \cdots \\
\hline
 & & & \frac{2}{15}z^5 & + \cdots
\end{array}$$

so $\boxed{\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots}$

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(20) Use $(\cos z)(\tan z) = \sin z$ to get

$$\left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots\right)(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots) = z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots$$

so $\underline{a_0 = 0}$, $\underline{a_1 = 1}$, $\underline{a_2 = 0}$, $-\frac{1}{2} + a_3 = -\frac{1}{6}$ so $\underline{a_3 = \frac{1}{3}}$,

$\underline{a_4 = 0}$, $a_5 - \frac{1}{6} + \frac{1}{24} = \frac{1}{120}$ so $\underline{a_5} = \dfrac{20 - 5 + 1}{120} = \dfrac{16}{120} = \underline{\dfrac{2}{15}}$

and $\boxed{\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots}$
