(4) $\dfrac{1}{z(z-1)(z-2)}$

a) $\underline{0<|z|<1}$

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{z}\left[\frac{1}{z-2} - \frac{1}{z-1}\right] = \frac{1}{z}\left[-\frac{1}{2-z} + \frac{1}{1-z}\right] = \frac{1}{z}\left[\frac{-1/2}{1-z/2} + \frac{1}{1-z}\right]$$

$$= \frac{1}{z}\left[\sum_{n=0}^{\infty} -\frac{1}{2}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} z^n\right] = \frac{1}{z}\sum_{n=0}^{\infty}\left[-\frac{1}{2^{n+1}} + 1\right] z^n$$

$$= \boxed{\sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right) z^{n-1}} = \boxed{\frac{1}{2z} + \frac{3}{4} + \frac{7}{8}z + \frac{15}{16}z^2 + \cdots}$$

b) $\underline{1<|z|<2}$

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{z}\left[\frac{1}{z-2} - \frac{1}{z-1}\right] = \frac{1}{z}\left[-\frac{1}{2-z} - \frac{1}{z-1}\right] = \frac{1}{z}\left[\frac{-1/2}{1-z/2} - \frac{1/z}{1-1/z}\right]$$

$$= \frac{1}{z}\left[\sum_{n=0}^{\infty} -\frac{1}{2}\left(\frac{z}{2}\right)^n - \sum_{n=0}^{\infty}\frac{1}{z}\left(\frac{1}{z}\right)^n\right] = \frac{1}{z}\left[\sum_{n=0}^{\infty} -\frac{1}{2^{n+1}}z^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right]$$

$$= \sum_{n=0}^{\infty} -\frac{1}{2^{n+1}}z^{n-1} - \sum_{n=0}^{\infty}\frac{1}{z^{n+2}}$$

$$= \boxed{-\sum_{n=2}^{\infty}\frac{1}{z^n} - \frac{1}{2z} + \sum_{n=0}^{\infty} -\frac{1}{2^{n+2}}z^n} = \boxed{\cdots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{2z} - \frac{1}{4} - \frac{1}{8}z - \frac{1}{16}z^2 - \cdots}$$

(7) Since f has a pole of order k at z_0,

$f(z) = \dfrac{\phi(z)}{(z-z_0)^k}$ where ϕ is analytic at z_0 and $\phi(z_0) \neq 0$.

Then $\displaystyle\lim_{z\to\infty}|f(z)| = \lim_{z\to\infty}\frac{|\phi(z)|}{|z-z_0|^k} = \infty$ since $|\phi(z)| \to |\phi(z_0)| \neq 0$ and $|z-z_0|^k \to 0^+$ as $z \to z_0$,

so $\displaystyle\lim_{z\to\infty} f(z) = \infty$.

(8) Let $f(z) = a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + \cdots$ and

$g(z) = c_k(z-z_0)^k + c_{k+1}(z-z_0)^{k+1} + \cdots$ where $a_k, c_k \neq 0$.

Then $$\frac{f(z)}{g(z)} = \frac{(z-z_0)^k\left[a_k + a_{k+1}(z-z_0) + \cdots\right]}{(z-z_0)^k\left[c_k + c_{k+1}(z-z_0) + \cdots\right]} = \frac{a_k + a_{k+1}(z-z_0) + \cdots}{c_k + c_{k+1}(z-z_0) + \cdots}$$

so $$\underline{\lim_{z\to z_0}\frac{f(z)}{g(z)}} = \frac{a_k}{c_k} = \frac{f^{(k)}(z_0)/k!}{g^{(k)}(z_0)/k!} = \underline{\frac{f^{(k)}(z_0)}{g^{(k)}(z_0)}},$$

and $\dfrac{f(z)}{g(z)}$ has a removable singularity at z_0 since this limit exists.

MATH 185A SOLUTIONS – SEC. 3.3 AND CH. 3 RE

(9) a) $\lim_{z\to 0} \dfrac{\cos(z-1)}{z^2}$ DOES NOT EXIST, SINCE $\lim_{z\to 0} \cos(z-1) = \cos(-1) \neq 0$ AND $\lim_{z\to 0} z^2 = 0$; SO $\dfrac{\cos(z-1)}{z^2}$ DOES NOT HAVE A REMOVABLE SING. AT $z_0 = 0$.

b) $\lim_{z\to 1} \dfrac{z}{z-1}$ DOES NOT EXIST, SINCE $z \to 1$ AND $z-1 \to 0$ AS $z \to 1$; SO $\dfrac{z}{z-1}$ DOES NOT HAVE A REMOVABLE SING. AT $z_0 = 1$.

c) $\dfrac{f(z)}{(z-z_0)^k}$ HAS A REMOVABLE SING. AT z_0 IF f HAS A ZERO OF ORDER k AT z_0, BY #8.

(11) $e^z - 1 = z + \dfrac{z^2}{2} + \dfrac{z^3}{6} + \dfrac{z^4}{24} + \cdots$, SO USING DIVISION TO FIND $\dfrac{1}{e^z-1}$ GIVES

$$
\begin{array}{r|l}
 & \frac{1}{z} - \frac{1}{2} + \frac{z}{12} - \frac{z^3}{720} + \cdots \\
\hline
z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots & 1 \\
 & 1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \frac{z^4}{120} + \cdots \\
\cline{2-2}
 & -\frac{z}{2} - \frac{z^2}{6} - \frac{z^3}{24} - \frac{z^4}{120} - \cdots \\
 & -\frac{z}{2} - \frac{z^2}{4} - \frac{z^3}{12} - \frac{z^4}{48} - \cdots \\
\cline{2-2}
 & \frac{z^2}{12} + \frac{z^3}{24} + \frac{z^4}{80} + \cdots \\
 & \frac{z^2}{12} + \frac{z^3}{24} + \frac{z^4}{72} + \cdots \\
\cline{2-2}
 & -\frac{z^4}{720} - \cdots
\end{array}
$$

SO $\dfrac{1}{e^z-1} = \boxed{\dfrac{1}{z} - \dfrac{1}{2} + \dfrac{z}{12} - \dfrac{z^3}{720} + \cdots}$

(16) SINCE $f(z) = (z-z_0)^k \phi(z)$ WHERE $\phi(z_0) \neq 0$,

$f'(z) = (z-z_0)^k \phi'(z) + k(z-z_0)^{k-1}\phi(z)$ AND THEREFORE

$$\frac{f'(z)}{f(z)} = \frac{(z-z_0)^{k-1}\left[(z-z_0)\phi'(z) + k\phi(z)\right]}{(z-z_0)^k \phi(z)} = \underline{\frac{\phi'(z)}{\phi(z)} + \frac{k}{z-z_0}}$$

WHERE $\dfrac{\phi'}{\phi}$ IS ANALYTIC AT z_0 (ASSUMING ϕ IS).

THEREFORE $\operatorname{Res}\left(\dfrac{f'}{f}; z_0\right) = k$.

(19) b) $\dfrac{z}{z^2-1}$, $z_0 = 1$ $\quad \lim_{z\to 1} (z-1)\left(\dfrac{z}{z^2-1}\right) = \lim_{z\to 1} \dfrac{z}{z+1} = \dfrac{1}{2} \neq 0$, SO $b_1 = \boxed{\tfrac{1}{2}}$

c) $\dfrac{e^z-1}{z^2}$, $z_0 = 0$ $\quad \dfrac{e^z-1}{z^2} = \dfrac{z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots}{z^2} = \dfrac{1}{z} + \dfrac{1}{2} + \dfrac{z}{6} + \cdots$, SO $\boxed{b_1 = 1}$

CH. 3 RE – (14) $f(z) = \dfrac{e^z}{(z-1)(z+1)(z-2)(z-3)}$, $z_0 = i$

THE DISTANCE FROM z_0 TO THE NEAREST SINGULARITY WILL GIVE THE RADIUS OF CONVERGENCE, SO $\boxed{R = \sqrt{2}}$