§8.2 Integration by Parts — Math 21B W21

- Integration by parts is the product rule run backwards.

Idea: Say we have the product of 2 functions —

$$f(x) = x$$

$$g(x) = e^x$$

$$f(x)g(x) = xe^x$$

$$\frac{d}{dx}\left(f(x)g(x)\right) = \frac{d}{dx}(xe^x) = e^x + xe^x$$

By FTC

$$\int \frac{d}{dx}\left(f(x)\,g(x)\right) dx = f(x)g(x) + \text{const}$$

$$\int f'(x)g(x) + f(x)g'(x)\, dx = f(x)\,g(x) + \text{Const}$$

~~In example~~ $\int$

So

$$\int f(x)g'(x)\,dx = -\int f'(x)\,g(x)\,dx + f(x)g(x)$$

$$\cancel{+C}$$

In our example $f(x) = x \quad g(x) = e^x$

$$\underbrace{\int x e^x\,dx}_{\text{not so}} = -\int 1 \cdot e^x + x e^x$$

$$= -e^x + xe^x + \text{const}$$

easy to do directly

Similarly: $$\int_a^b f(x)g'(x)\,dx = -\int_a^b f'(x)g(x)\,dx + f(x)g(x)\Big]_a^b$$

Another way to view it:

$$\int_a^b x e^x \, dx = \int_a^b \frac{d}{dx}(xe^x) - e^x \, dx$$

$$= xe^x \Big]_a^b - e^x \Big]_a^b$$

$$\int_a^b f(x) g'(x) \, dx = \int_a^b \frac{d}{dx}\left(f(x) g(x)\right) - f'(x) g(x) \, dx$$

$$= fg \Big]_a^b - \int f'(x) g(x) \, dx$$

✓

Notation for doing this —

$$\int \underbrace{f(x)}_{u} \underbrace{g'(x)\,dx}_{dv} = \underbrace{fg}_{uv} - \int \underbrace{g}_{v} \underbrace{f'\,dx}_{du}$$

$$\boxed{\int u \, dv = uv - \int v \, du}$$

Ex 1 $\int x \sin x \, dx = \int u \, dv$

Soln: $u = x \quad dv = \sin x \, dx$

$du = dx \quad v = -\cos x$

$= vu - \int v \, du = -x\cos x + \int \cos x \, dx$

$= -x\cos x + \sin x + C$

$\int_0^{2\pi} x \sin x = uv\Big]_0^{2\pi} - \int_0^{2\pi} v \, du$

$= -x\cos x\Big]_0^{2\pi} + \cancel{\sin x\Big]_0^{2\pi}}$ 0

$= -2\pi + 0 = \boxed{-2\pi}$

Ex 2 $\int \ln x \, dx = \int u \, dv$

$$u = \ln x \quad dv = dx$$

$$du = \frac{1}{x} dx \quad v = x$$

$$= vu - \int v \, du = x \ln x - \int x \cdot \frac{1}{x} \, dx$$

$$= x \ln x - x + C$$

Check: $\frac{d}{dx}(x \ln x - x + C) = \ln x + \frac{x}{x} - 1$

$$= \ln x$$

$$\int_1^2 \ln x = uv \Big]_1^2 - \int_1^2 v \, du$$

$$= x \ln x \Big]_1^2 - \int_1^2 x \frac{1}{x} \, dx$$

$$= 2 \ln 2 - \underbrace{\cancel{\ln 1}}_{0} - 1 = \boxed{2 \ln 2 - 1}$$

Ex 3 $\int x^2 e^x \, dx = \int u \, dv$

$$u = x^2 \qquad dv = e^x dx$$
$$du = 2x \, dx \qquad v = e^x$$

$$= uv - \int v \, du = x^2 e^x - \underbrace{\int 2x e^x dx}_{\text{int by parts again}}$$

$$+\int 2x e^x dx = 2xe^x - \int 2e^x dx$$
$$= 2xe^x - 2e^x$$

$$u = 2x \qquad dv = e^x dx$$
$$du = 2dx \qquad v = e^x$$

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C \checkmark$$

Note: Same idea $\int x^{10} \sin 3x \, dx$

10 integration by parts !

$$\int x^{25} e^x dx$$

Trig Integrals : $\int \sin^m x \cos^n x \, dx$

(I) If m or n is odd: say

$$\int \sin^4 x \cos^3 x$$

$$= \int \sin^4 x \underbrace{\cos^2 x}_{1-\sin^2 x} \cos x \, dx$$

$$= \int \sin^4 x \left(1-\sin^2 x\right) \cos x \, dx$$

$u = \sin x \quad du = \cos x \, dx$

$$= \int u^4 (1-u^2) \, du = \frac{u^5}{5} - \frac{u^7}{7} + C$$

$$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$$

If both m,n even:

$$\int \cos^2 x \sin^2 x \, dx$$

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$= \int \frac{1}{4}(1-\cos 2x)(1+\cos 2x)\, dx$$

$$= \frac{1}{4}\int 1-\cos^2 2x \, dx = \frac{1}{4}x - \frac{1}{4}\int \cos^2 2x \, dx$$

$$\cos^2 2x = \frac{1}{2}(1+\cos 4x)$$

$$= \frac{1}{4}x - \frac{1}{4}\int \frac{1}{2}(1+\cos 4x)\, dx$$

$$= \frac{1}{4}x - \frac{1}{4}\left(\frac{1}{2}x + \frac{1}{4}\sin 4x\right) + C$$

Ex: $\int \sin(mx)\cos(nx)\,dx$ (F-series! Important)

Use Trig Sum Angle Identities –

$$\sin(mx)\sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

$$\sin(mx)\cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right.$$

$$\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$$

Ie. $\sin(m \pm n)x = \sin mx \cos nx \pm \sin nx \cos mx$

$\cos(m \pm n)x = \cos mx \cos nx \mp \sin mx \cos nx$