Section 10.1

2.) $a_n = \frac{1}{n!}$ so $a_1 = \frac{1}{1!} = 1$, $a_2 = \frac{1}{2!} = \frac{1}{2}$,
$a_3 = \frac{1}{3!} = \frac{1}{6}$, $a_4 = \frac{1}{4!} = \frac{1}{24}$

3.) $a_n = \frac{(-1)^{n+1}}{2n-1}$ so $a_1 = \frac{(-1)^2}{1} = 1$,
$a_2 = \frac{(-1)^3}{3} = \frac{-1}{3}$, $a_3 = \frac{(-1)^4}{5} = \frac{1}{5}$,
$a_4 = \frac{(-1)^5}{7} = \frac{-1}{7}$

4.) $a_n = 2 + (-1)^n$ so $a_1 = 2 + (-1) = 1$,
$a_2 = 2 + (-1)^2 = 2 + 1 = 3$,
$a_3 = 2 + (-1)^3 = 2 - 1 = 1$,
$a_4 = 2 + (-1)^4 = 2 + 1 = 3$.

8.) $a_1 = 1$, $a_{n+1} = \frac{a_n}{n+1}$ so
$a_2 = \frac{a_1}{2} = \frac{1}{2}$, $a_3 = \frac{a_2}{3} = \frac{1}{6} = \frac{1}{3!}$,
$a_4 = \frac{a_3}{4} = \frac{1}{24} = \frac{1}{4!}$, $a_5 = \frac{a_4}{5} = \frac{1}{5!}$,
$a_6 = \frac{1}{6!}$, $a_7 = \frac{1}{7!}$, $a_8 = \frac{1}{8!}$, $a_9 = \frac{1}{9!}$,
$a_{10} = \frac{1}{10!}$

11.) $a_1 = a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$ so
$a_3 = a_2 + a_1 = 1 + 1 = 2$,
$a_4 = a_3 + a_2 = 2 + 1 = 3$,
$a_5 = a_4 + a_3 = 3 + 2 = 5$,
$a_6 = a_5 + a_4 = 5 + 3 = 8$,

$a_7 = a_6 + a_5 = 8+5 = 13$,
$a_8 = a_7 + a_6 = 13+8 = 21$,
$a_9 = a_8 + a_7 = 21+13 = 34$,
$a_{10} = a_9 + a_8$

12.) $a_1 = 2$, $a_2 = -1$, $a_{n+2} = \dfrac{a_{n+1}}{a_n}$ so

$$a_3 = \frac{a_2}{a_1} = \frac{-1}{2},$$

$$a_4 = \frac{a_3}{a_2} = \frac{-1/2}{-1} = \frac{1}{2},$$

$$a_5 = \frac{a_4}{a_3} = \frac{1/2}{-1/2} = -1,$$

$$a_6 = \frac{a_5}{a_4} = \frac{-1}{1/2} = -2$$

$$a_7 = \frac{a_6}{a_5} = \frac{-2}{-1} = 2,$$

$$a_8 = \frac{a_7}{a_6} = \frac{2}{-2} = -1,$$

$$a_9 = \frac{a_8}{a_7} = \frac{-1}{2},$$

$$a_{10} = \frac{a_9}{a_8} = \frac{-1/2}{-1} = \frac{1}{2}$$

13.) $(-1)^{n+1}$ for $n = 1, 2, 3, \ldots$

16.) $(-1)^{n+1} \cdot \frac{1}{n^2}$ for $n = 1, 2, 3, \ldots$

18.) $\dfrac{-5+2n}{n(n+1)}$ for $n = 1, 2, 3, \ldots$

19.) $(n-1)^2$ for $n = 1, 2, 3, \ldots$

21.) $4n-3$ for $n = 1, 2, 3, \ldots$

23.) $\dfrac{3n+2}{n!}$ for $n = 1, 2, 3, \ldots$

24.) $\dfrac{n^3}{5^n}$ for $n = 1, 2, 3, \ldots$

~~25.) $\frac{1}{2} + \frac{1}{2}(-1)^{n+1}$ for $n = 1, 2, 3, \ldots$~~

26.) $\left\lfloor \dfrac{n}{2} \right\rfloor$ for $n = 1, 2, 3, \ldots$

28.) $-1 \le (-1)^n \le +1 \rightarrow$

$n-1 \le n+(-1)^n \le n+1 \rightarrow$

$\dfrac{n-1}{n} \le \dfrac{n+(-1)^n}{n} \le \dfrac{n+1}{n}$; then

$\lim\limits_{n\to\infty} \dfrac{n-1}{n} = \lim\limits_{n\to\infty} \left(1 - \dfrac{1}{n}\right) = 1 - 0 = 1$ and

$\lim\limits_{n\to\infty} \dfrac{n+1}{n} = \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right) = 1 + 0 = 1$ so

by Sandwich Theorem $\lim\limits_{n\to\infty} \dfrac{n+(-1)^n}{n} = 1$.

32.) $\lim\limits_{n\to\infty} \dfrac{n+3}{n^2+5n+6} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim\limits_{n\to\infty} \dfrac{1}{2n+5} = \text{"}\dfrac{1}{\infty}\text{"} = 0$

33.) $\lim\limits_{n\to\infty} \dfrac{n^2-2n+1}{n-1} = \lim\limits_{n\to\infty} \dfrac{(n-1)^2}{n-1}$

$= \lim\limits_{n\to\infty} (n-1) = \infty$ (diverges)

35.) $a_n = 1+(-1)^n$: 0, 2, 0, 2, 0, 2, ... so

$\lim_{n\to\infty} a_n$ DNE (by oscillation)

39.) $-1 \le (-1)^{n+1} \le +1 \longrightarrow$

$\frac{-1}{2n-1} \le \frac{(-1)^{n+1}}{2n-1} \le \frac{1}{2n-1}$; then

$\lim_{n\to\infty} \frac{-1}{2n-1} = "\frac{-1}{\infty}" = 0$ and

$\lim_{n\to\infty} \frac{1}{2n-1} = "\frac{1}{\infty}" = 0$ so by

Sandwich Theorem $\lim_{n\to\infty} \frac{(-1)^{n+1}}{2n-1} = 0$.

40.) $\lim_{n\to\infty} \left(\frac{-1}{2}\right)^n = 0$ since $-1 < \frac{-1}{2} < 1$

42.) $\lim_{n\to\infty} \frac{1}{(0.9)^n} = \lim_{n\to\infty} \frac{1}{\frac{9^n}{10^n}}$

$= \lim_{n\to\infty} \frac{10^n}{9^n} = \lim_{n\to\infty} \left(\frac{10}{9}\right)^n = \infty$ (diverges)

since $\frac{10}{9} > 1$.

43.) $\lim_{n\to\infty} \sin\left(\frac{\pi}{2} + \frac{1}{n}\right) = \sin\left(\frac{\pi}{2} + 0\right)$

$= \sin\frac{\pi}{2} = 1$

46.) $-1 \le \sin n \le +1 \rightarrow 0 \le \sin^2 n \le 1 \longrightarrow$

$\frac{0}{2^n} \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$; then

$\lim_{n\to\infty} \frac{0}{2^n} = \lim_{n\to\infty} 0 = 0$ and

$\lim_{n\to\infty} \frac{1}{2^n} = "\frac{1}{\infty}" = 0$ so by Sandwich

Theorem $\lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0$

48.) $\lim_{n\to\infty} \frac{3^n}{n^3} \overset{"\frac{\infty}{\infty}"}{=} \lim_{n\to\infty} \frac{3^n \cdot \ln 3}{3n^2} \overset{"\frac{\infty}{\infty}"}{=}$

$\lim_{n\to\infty} \frac{3^n \cdot (\ln 3)^2}{6n} \overset{"\frac{\infty}{\infty}"}{=} \lim_{n\to\infty} \frac{3^n \cdot (\ln 3)^3}{6} = \infty$

(diverges)

49.) $\lim_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}} \overset{"\frac{\infty}{\infty}"}{=} \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{2\sqrt{n}}}$

$= \lim_{n\to\infty} \frac{2\sqrt{n}}{n+1} \overset{"\frac{\infty}{\infty}"}{=} \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{1} = "\frac{1}{\infty}" = 0$

52.) $\lim_{n\to\infty} (0.03)^{1/n} = (0.03)^0 = 1$

54.) $\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{(-1)}{n}\right)^n = e^{-1}$

56.) $\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} (n^2)^{1/n} = \lim_{n\to\infty} n^{2/n}$

$"\infty^0"$

$= \lim_{n\to\infty} e^{\ln n^{2/n}} = \lim_{n\to\infty} e^{\frac{2}{n}\ln n}$

$= \lim_{n\to\infty} e^{\frac{2\ln n}{n}} \overset{"\frac{\infty}{\infty}"}{=} \lim_{n\to\infty} e^{\frac{2\cdot\frac{1}{n}}{1}} = e^{2\cdot(0)} = 1$

57.) $\lim_{n\to\infty} \left(\frac{3}{n}\right)^{1/n} \overset{"0^0"}{=} \lim_{n\to\infty} e^{\ln\left(\frac{3}{n}\right)^{1/n}}$

$= \lim_{n\to\infty} e^{\frac{1}{n}\ln\left(\frac{3}{n}\right)} = \lim_{n\to\infty} e^{\frac{\ln(3/n)}{n}}$

$\overset{"\frac{\infty}{\infty}"}{=} \lim_{n\to\infty} e^{\frac{\frac{1}{3/n}\cdot \frac{-3}{n^2}}{1}} = \lim_{n\to\infty} e^{-1/n} = e^0 = 1$

59.) $\lim_{n\to\infty} n^{1/n} \overset{"\infty^0"}{=} \lim_{n\to\infty} e^{\ln n^{1/n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n}$

$= \lim_{n\to\infty} e^{\frac{\ln n}{n}} \overset{"\frac{\infty}{\infty}"}{=} \lim_{n\to\infty} e^{\frac{1/n}{1}} = e^0 = 1$; THEN

$\lim_{n\to\infty} \frac{\ln n}{n^{1/n}} = \frac{\infty}{1} = \infty$

60.) $\lim_{n\to\infty} (\ln n - \ln(n+1)) \overset{"\infty-\infty"}{=} \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right)$

$= \ln\left(\lim_{n\to\infty} \frac{n}{n+1}\right) \overset{"\frac{\infty}{\infty}"}{=} \ln\left(\lim_{n\to\infty} \frac{1}{1}\right) = \ln 1 = 0$

by continuity of $y = \ln x$

63.) $a_n = \frac{n!}{n^n}$

$$n: \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \ldots$$

$$\frac{n!}{n^n}: \quad \frac{1}{1}, \frac{2\cdot 1}{2\cdot 2}, \frac{3\cdot 2\cdot 1}{3\cdot 3\cdot 3}, \frac{4\cdot 3\cdot 2\cdot 1}{4\cdot 4\cdot 4\cdot 4}, \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{5\cdot 5\cdot 5\cdot 5\cdot 5}, \ldots$$

$$= 1, \frac{1}{2}, \left(\frac{2}{3}\right)\left(\frac{1}{3}\right), \left(\frac{3}{4}\right)\left(\frac{2}{4}\right)\left(\frac{1}{4}\right), \left(\frac{4}{5}\right)\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)\left(\frac{1}{5}\right), \ldots$$

$$\leq 1, \frac{1}{2}, (1)\left(\frac{1}{3}\right), (1)(1)\left(\frac{1}{4}\right), (1)(1)(1)\left(\frac{1}{5}\right), \ldots$$

$$\leq 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \text{ i.e.,}$$

$0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$; then

$\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{1}{n}$ so by

Sandwich Theorem $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

64.) $\lim_{n\to\infty} \frac{(-4)^n}{n!} = 0$ since

<u>RULE</u>: $\lim_{n\to\infty} \frac{k^n}{n!} = 0$ for any constant k

66.) $\lim_{n\to\infty} \frac{n!}{2^n 3^n} = \lim_{n\to\infty} \frac{n!}{(2\cdot 3)^n} = \lim_{n\to\infty} \frac{n!}{6^n}$

$= \lim_{n\to\infty} \frac{1}{\frac{6^n}{n!}} = \frac{1}{0^+} = +\infty$

by RULE

67.) $\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/\ln n} \overset{\text{"}0^0\text{"}}{=} \lim_{n\to\infty} e^{\ln\left(\frac{1}{n}\right)^{1/\ln n}}$

$$= \lim_{n\to\infty} e^{\frac{1}{\ln n}\cdot \ln(1/n)}$$

$$= e^{\lim\limits_{n\to\infty} \frac{\ln(1/n)}{\ln n}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} e^{\lim\limits_{n\to\infty} \frac{\frac{1}{1/n}\cdot\frac{-1}{n^2}}{1/n}}$$

$$= e^{\lim\limits_{n\to\infty} \frac{-1/n}{1/n}} = e^{\lim\limits_{n\to\infty}(-1)} = e^{-1}$$

68.) $\lim\limits_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^{n} \overset{\text{"}1^\infty\text{"}}{=} \lim\limits_{n\to\infty} n\cdot \ln(1+1/n)$

$$\overset{\text{"}\infty\cdot 0\text{"}}{=} \lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{1+1/n}\cdot \cancel{\frac{-1}{n^2}}}{\cancel{-1/n^2}}$$

$$= \frac{1}{1+0} = 1$$

69.) $\lim\limits_{n\to\infty}\left(\frac{3n+1}{3n-1}\right)^{n} = \lim\limits_{n\to\infty}\left(\frac{3n-1+2}{3n-1}\right)^{n}$

$$= \lim_{n\to\infty}\left(\frac{3n-1}{3n-1}+\frac{2}{3n-1}\right)^{n}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{\frac{3n-1}{2}}\right)^{\frac{3n-1}{2}\cdot\frac{2}{3n-1}\cdot n}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{3n-1}{2}}\right)^{\frac{3n-1}{2}}\right]^{\frac{2n}{3n-1}} \quad \text{"}\frac{\infty}{\infty}\text{"}$$

$$= [e]^{2/3}$$

74.) $\lim\limits_{n\to\infty} \dfrac{\left(\frac{10}{11}\right)^n}{\left(\frac{9}{10}\right)^n + \left(\frac{11}{12}\right)^n}$

$= \lim\limits_{n\to\infty} \dfrac{\left(\frac{10}{11}\right)^n}{\left(\frac{9}{10}\right)^n + \left(\frac{11}{12}\right)^n} \cdot \dfrac{\frac{1}{(11/12)^n}}{\frac{1}{\left(\frac{11}{12}\right)^n}}$

$= \lim\limits_{n\to\infty} \dfrac{\left(\frac{120}{121}\right)^n}{\left(\frac{108}{110}\right)^n + 1} = \dfrac{0}{0+1} = 0$

78.) $\lim\limits_{n\to\infty} n\left(1 - \cos\frac{1}{n}\right) = \lim\limits_{n\to\infty} \dfrac{1-\cos\frac{1}{n}}{\frac{1}{n}}$

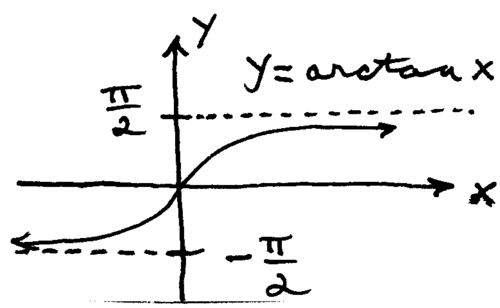
$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{n\to\infty} \dfrac{\sin\frac{1}{n} \cdot \cancel{\frac{-1}{n^2}}}{\cancel{\frac{-1}{n^2}}} = \sin 0 = 0$

79.) $\lim\limits_{n\to\infty} \sqrt{n}\, \sin\frac{1}{\sqrt{n}} \overset{\text{"}\infty\cdot 0\text{"}}{=} \lim\limits_{n\to\infty} \dfrac{\sin\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} \overset{\text{"}\frac{0}{0}\text{"}}{=} 1$

80.) $\lim\limits_{n\to\infty} \left(3^n + 5^n\right)^{1/n} \overset{\text{"}\infty^0\text{"}}{=} \lim\limits_{n\to\infty} \left[5^n\left(\frac{3^n}{5^n} + 1\right)\right]^{1/n}$

$= \lim\limits_{n\to\infty} 5 \cdot \left[\left(\frac{3}{5}\right)^n + 1\right] = 5 \cdot \left[(0) + 1\right] = 5$

81.) $\lim\limits_{n\to\infty} \arctan n = \frac{\pi}{2}$



83.) $\lim\limits_{n\to\infty} \left[\left(\frac{1}{3}\right)^n + \frac{1}{2^{n/2}}\right] = 0 + 0 = 0$

86.) $\lim\limits_{n\to\infty} \dfrac{(\ln n)^5}{\sqrt{n}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim\limits_{n\to\infty} \dfrac{5(\ln n)^4 \cdot 1/n}{1/2\sqrt{n}}$

$$= \lim_{n\to\infty} \frac{10(\ln n)^4}{\sqrt{n}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{40(\ln n)^3 \cdot 1/n}{1/2\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{80(\ln n)^3}{\sqrt{n}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{240(\ln n)^2 \cdot 1/n}{1/2\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{480(\ln n)^2}{\sqrt{n}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{960(\ln n) \cdot 1/n}{1/2\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1920 \ln n}{\sqrt{n}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{1920 \cdot 1/n}{1/2\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{3840}{\sqrt{n}} = 0$$

87.) $\lim_{n\to\infty} (n - \sqrt{n^2 - n}) = \text{"}\infty - \infty\text{"}$

$$= \lim_{n\to\infty} (n - \sqrt{n^2 - n}) \frac{(n + \sqrt{n^2 - n})}{(n + \sqrt{n^2 - n})}$$

$$= \lim_{n\to\infty} \frac{n^2 - (n^2 - n)}{n + \sqrt{n^2 - n}} = \lim_{n\to\infty} \frac{n}{n + \sqrt{n^2(1 - \frac{1}{n})}}$$

$$= \lim_{n\to\infty} \frac{n}{n + n\sqrt{1 - \frac{1}{n}}} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1 + \sqrt{1 - \frac{1}{n}}} = \frac{1}{1 + \sqrt{1 - 0}} = 1/2$$

89.) $\lim_{n\to\infty} \frac{1}{n} \int_1^n \frac{1}{x}\,dx = \lim_{n\to\infty} \frac{\int_1^n \frac{1}{x}\,dx}{n}$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0$$

92.) $a_{n+1} = \frac{a_n + 6}{a_n + 2}$; Let limit

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L :$$

$\rightarrow L = \frac{L+6}{L+2} \rightarrow L(L+2) = L+6$

$\rightarrow L^2 + 2L = L+6 \rightarrow L^2 + L - 6 = 0 \rightarrow$

$(L-2)(L+3) = 0 \rightarrow \underline{L=2}$ or $\underline{L=-3}$;

BUT $a_1 = -1$ so

$a_2 = \frac{(-1)+6}{(-1)+2} = 5$, $a_3 = \frac{(5)+6}{(5)+2} = \frac{11}{7}, \ldots$;

CLEARLY all terms of this sequence are POSITIVE, so the limit is $\boxed{L=2}$.

93.) $a_{n+1} = \sqrt{8+2a_n}$ (Let L be limit)

$\rightarrow L = \sqrt{8+2L} \rightarrow L^2 = 8+2L \rightarrow$

$L^2 - 2L - 8 = 0 \rightarrow (L-4)(L+2) = 0$

$\rightarrow \underline{L=4}$ or $\underline{L=-2}$; BUT $\sqrt{8+2a_n}$ is always positive, so the limit is $\boxed{L=4}$.

96.) $a_{n+1} = 12 - \sqrt{a_n}$ (Let L be limit)

$\rightarrow L = 12 - \sqrt{L} \rightarrow \sqrt{L} = 12 - L$

$\rightarrow (\sqrt{L})^2 = (12-L)^2 \rightarrow L = 144 - 24L + L^2$

$\rightarrow 0 = L^2 - 25L + 144 = (L-16)(L-9)$

$\rightarrow \underline{L=16}$ or $\underline{L=9}$; BUT $a_1 = 3$ so

$a_2 = 12 - \sqrt{3} < 12$ so $a_{n+1} = 12 - \sqrt{a_n}$ < 12, so the limit is $\boxed{L=9}$.

97.) $$2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}} = \overset{\text{limit}}{L}$$

$$\rightarrow 2+\frac{1}{L} = L \quad (\text{mult. by } L)$$

$$\rightarrow 2L+1 = L^2 \rightarrow L^2-2L-1=0$$

$$\Rightarrow L = \frac{-(-2)\pm\sqrt{4-4(1)(-1)}}{2(1)} = \frac{2\pm2\sqrt{2}}{2}$$

$= 1\pm\sqrt{2}$; BUT clearly $L>0$,

so limit is $\boxed{L=1+\sqrt{2}}$

98.) Let L be limit of this sequence:

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}} = L \rightarrow$$

$$\sqrt{1+L} = L \rightarrow \left(\sqrt{1+L}\right)^2 = L^2 \rightarrow$$

$$1+L = L^2 \rightarrow 0 = L^2-L-1 \rightarrow$$

$$L = \frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)} = \frac{-1\pm\sqrt{5}}{2} \rightarrow$$

$$L = \frac{\sqrt{5}-1}{2}.$$

99.) $x_1 = 1$, $x_{n+1} = x_1 + x_2 + x_3 + \cdots + x_n$, then

$x_2 = x_1 = 1$,

$x_3 = x_1 + x_2 = 1+1 = 2$,

$x_4 = x_1 + x_2 + x_3 = 1+1+2 = 4$,

$x_5 = x_1 + x_2 + x_3 + x_4 = 1+1+2+4 = 8$,

$x_6 = x_1 + x_2 + x_3 + x_4 + x_5 = 1+1+2+4+8 = 16$,

$x_7 = \ldots = 32$, $x_8 = 64$, ...

$x_n = 2^{n-2}$ for $n = 2, 3, 4, 5, \ldots$

108.) Prove that $\lim_{n \to \infty} x^{1/n} = 1$ (for $x > 1$):

Let $\varepsilon > 0$ be given. Find integer N so that if $n > N$, then $|x^{1/n} - 1| < \varepsilon$.

Then $|x^{1/n} - 1| < \varepsilon$ iff $x^{1/n} - 1 < \varepsilon$ $\left(\begin{matrix}\text{since}\\ x>1\end{matrix}\right)$

iff $x^{1/n} < \varepsilon + 1$

iff $\ln x^{1/n} < \ln(\varepsilon + 1)$

iff $\frac{1}{n} \ln x < \ln(\varepsilon + 1)$

iff $n > \frac{\ln x}{\ln(\varepsilon+1)}$. Let N be any integer $\geq \frac{\ln x}{\ln(\varepsilon+1)}$. Thus, if $n > N$, then $|x^{1/n} - 1| < \varepsilon$.

QED

116.) $\lim\limits_{n\to\infty} \left(n - \frac{1}{n}\right) = \infty - 0 = \infty$ (diverges)

118.) $\lim\limits_{n\to\infty} \dfrac{2^n - 1}{3^n} = \lim\limits_{n\to\infty} \left(\dfrac{2^n}{3^n} - \dfrac{1}{3^n}\right)$

$= \lim\limits_{n\to\infty} \left(\left(\frac{2}{3}\right)^n - \dfrac{1}{3^n}\right) = 0 - 0 = 0$ (converges)

119.) $\lim\limits_{n\to\infty} \left((-1)^n + 1\right)\left(\dfrac{n+1}{n}\right) = \lim\limits_{n\to\infty} \left((-1)^n + 1\right)\left(1 + \dfrac{1}{n}\right)$

$= \begin{cases} \lim\limits_{n\to\infty} (2)\left(1 + \frac{1}{n}\right) = 2(1+0) = 2, & \text{~~if n even~~} \\ \lim\limits_{n\to\infty} (0)\left(1 + \frac{1}{n}\right) = 0, & \text{if n even,} \end{cases}$

so $\lim\limits_{n\to\infty} \left((-1)^n + 1\right)\left(\dfrac{n+1}{n}\right)$ DNE (diverges)

121.) $\lim\limits_{n\to\infty} \dfrac{1 + \sqrt{2n}}{\sqrt{n}} = \lim\limits_{n\to\infty} \dfrac{1 + \sqrt{2}\sqrt{n}}{\sqrt{n}}$

$= \lim\limits_{n\to\infty} \left(\dfrac{1}{\sqrt{n}} + \sqrt{2}\right) = 0 + \sqrt{2} = \sqrt{2}$

123.) $\lim\limits_{n\to\infty} \dfrac{4^{n+1} + 3^n}{4^n} = \lim\limits_{n\to\infty} \left(\dfrac{4^{n+1}}{4^n} + \dfrac{3^n}{4^n}\right)$

$= \lim\limits_{n\to\infty} \left(4 + \left(\frac{3}{4}\right)^n\right) = 4 + 0 = 4$

124.) $a_1 = 1,\ a_{n+1} = 2a_n - 3 \to a_2 = 2a_1 - 3$
$= 2(1) - 3 = -1,\ a_3 = 2a_2 - 3 = 2(-1) - 3 = -5,$
$a_4 = 2a_3 - 3 = 2(-5) - 3 = -13$, so clearly
$\lim\limits_{n\to\infty} a_n = -\infty$ (diverges)

Math 21C
Kouba
Worksheet 1

1.) Use the precise deifinition of the limit of a sequence to prove each of the following statements.

a.) $\lim_{n \to \infty} \frac{1}{n+5} = 0$

b.) $\lim_{n \to \infty} \frac{3}{\sqrt{n+2}} = 0$

c.) $\lim_{n \to \infty} \frac{n+3}{1-n} = -1$

d.) $\lim_{n \to \infty} (0.9)^n = 0$

e.) $\lim_{n \to \infty} 3(0.25)^{n-2} = 0$

Worksheet 1

1.) a.) Prove that $\lim_{n\to\infty} \frac{1}{n+5} = 0$:

Let $\varepsilon > 0$ be given. Find an integer N so that if $n > N$, then $\left|\frac{1}{n+5} - 0\right| < \varepsilon$. Then

$\left|\frac{1}{n+5} - 0\right| < \varepsilon$ iff $\frac{1}{n+5} < \varepsilon$ (assume $n > 0$)

iff $n + 5 > \frac{1}{\varepsilon}$

iff $n > \frac{1}{\varepsilon} - 5$. Choose any integer $N \geq \frac{1}{\varepsilon} - 5$. Thus, if $n > N$, then $\left|\frac{1}{n+5} - 0\right| < \varepsilon$.

QED

b.) Prove that $\lim_{n\to\infty} \frac{3}{\sqrt{n+2}} = 0$:

Let $\varepsilon > 0$ be given. Find an integer N so that if $n > N$, then $\left|\frac{3}{\sqrt{n+2}} - 0\right| < \varepsilon$. Then

$\left|\frac{3}{\sqrt{n+2}} - 0\right| < \varepsilon$ iff $\frac{3}{\sqrt{n+2}} < \varepsilon$

iff $\sqrt{n+2} > \frac{3}{\varepsilon}$

iff $n + 2 > \frac{9}{\varepsilon^2}$

iff $n > \frac{9}{\varepsilon^2} - 2$. Choose any integer $N \geq \frac{9}{\varepsilon^2} - 2$. Thus, if $n > N$, then $\left| \frac{3}{\sqrt{n+2}} - 0 \right| < \varepsilon$.

QED

c.) Prove that $\lim_{n \to \infty} \frac{n+3}{1-n} = -1$:

Let $\varepsilon > 0$ be given. Find integer N so that if $n > N$, then $\left| \frac{n+3}{1-n} - (-1) \right| < \varepsilon$. Then

$\left| \frac{n+3}{1-n} - (-1) \right| < \varepsilon$ iff $\left| \frac{n+3}{1-n} + \frac{1-n}{1-n} \right| < \varepsilon$

iff $\left| \frac{4}{1-n} \right| < \varepsilon$

iff $\frac{4}{-(1-n)} < \varepsilon$ (assume $n > 1$.)

iff $\frac{4}{n-1} < \varepsilon$

iff $n - 1 > \frac{4}{\varepsilon}$

iff $n > \frac{4}{\varepsilon} + 1$. Choose any integer $N \geq \frac{4}{\varepsilon} + 1$. Thus, if $n > N$, then $\left| \frac{n+3}{1-n} - (-1) \right| < \varepsilon$.

QED

d.) Prove that $\lim_{n\to\infty} (0.9)^n = 0$:

Let $\varepsilon > 0$ be given. Find integer N so that if $n > N$, then $|(0.9)^n - 0| < \varepsilon$. Then

$$|(0.9)^n| < \varepsilon \text{ iff } (0.9)^n < \varepsilon \quad (\text{since } (0.9)^n > 0)$$

$$\text{iff } \ln (0.9)^n < \ln \varepsilon$$

$$\text{iff } n(\ln(0.9)) < \ln \varepsilon$$

$$\text{iff } n > \frac{\ln \varepsilon}{\ln (0.9)} \quad (\text{since } \ln(0.9) < 0).$$

Choose any integer $N \geq \dfrac{\ln \varepsilon}{\ln(0.9)}$. Thus, if $n > N$, then $|(0.9)^n - 0| < \varepsilon$.

QED

e.) Prove that $\lim_{n\to\infty} 3(0.25)^{n-2} = 0$:

Let $\varepsilon > 0$ be given. Find integer N so that if $n > N$, then $|3(0.25)^{n-2} - 0| < \varepsilon$. Then

$$|3(0.25)^{n-2}| < \varepsilon \text{ iff } 3(0.25)^{n-2} < \varepsilon$$

$$(\text{since } 3(0.25)^{n-2} > 0)$$

iff $(0.25)^{n-2} < \varepsilon/3$

iff $\ln (0.25)^{n-2} < \ln(\varepsilon/3)$

iff $(n-2)\ln(0.25) < \ln(\varepsilon/3)$

iff $n-2 > \dfrac{\ln(\varepsilon/3)}{\ln(0.25)}$ (since $\ln(0.25) < 0$)

iff $n > 2 + \dfrac{\ln(\varepsilon/3)}{\ln(0.25)}$. Choose any integer $N \geq 2 + \dfrac{\ln(\varepsilon/3)}{\ln(0.25)}$.

Thus, if $n > N$, then

$|3(0.25)^{n-2} - 0| < \varepsilon$.

Q.E.D.