Section 10.10

1.) $(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \cdots$

$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$

2.) $(1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)}{2!}x^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}x^3 + \cdots$

$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + \cdots$

6.) $\left(1-\frac{x}{3}\right)^4 = \left(1+\left(\frac{-x}{3}\right)\right)^4$

$= 1 + 4\left(\frac{-x}{3}\right) + \frac{4(4-1)}{2!}\left(\frac{-x}{3}\right)^2 + \frac{4(4-1)(4-2)}{3!}\left(\frac{-x}{3}\right)^3$

$+ \frac{4(4-1)(4-2)(4-3)}{4!}\left(\frac{-x}{3}\right)^4$

$= 1 - \frac{4}{3}x + \frac{2}{3}x^2 - \frac{4}{27}x^3 + \frac{1}{81}x^4$ (finite length!)

7.) $(1+x^3)^{-1/2} = 1 + \frac{-1}{2}(x^3) + \frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)}{2!}(x^3)^2 + \frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}(x^3)^3 + \cdots$

$= 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \cdots$

10.) $\frac{x}{(1+x)^{1/3}} = x(1+x)^{-1/3}$

$= x\left[1 + \frac{-1}{3}x + \frac{\frac{-1}{3}\left(\frac{-1}{3}-1\right)}{2!}x^2 + \frac{\frac{-1}{3}\left(\frac{-1}{3}-1\right)\left(\frac{-1}{3}-2\right)}{3!}x^3 + \cdots\right]$

$$= x\left[1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \cdots\right]$$

$$= x - \frac{1}{3}x^2 + \frac{2}{9}x^3 - \frac{14}{81}x^4 + \cdots$$

12.) $(1+(x^2))^3 = 1 + 3(x^2) + \frac{3(3-1)}{2!}(x^2)^2 + \frac{3(3-1)(3-2)}{3!}(x^2)^3$

$= 1 + 3x^2 + 3x^4 + x^6$ (finite length!)

15.) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \longrightarrow$

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots$$

$$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots \longrightarrow$$

$$\int_0^{0.2} \sin(x^2)\,dx = \int_0^{0.2}\left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots\right)dx$$

$$= \left(\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots\right)\Big|_0^{0.2}$$

$$= \frac{(0.2)^3}{3} - \frac{(0.2)^7}{7\cdot 3!} + \frac{(0.2)^{11}}{11\cdot 5!} - \frac{(0.2)^{15}}{15\cdot 7!} + \cdots$$

$\approx 3\times 10^{-14} < 10^{-3}$, so by convergent alternating series facts,

$$\int_0^{0.2} \sin(x^2)\,dx \approx \frac{(0.2)^3}{3} \approx 0.00267$$

with absolute error

$$|R_1| \le \frac{(0.2)^7}{7\cdot 3!} \approx 3\times 10^{-14} < 10^{-3}.$$

17.) $\int_0^{0.1} \frac{1}{\sqrt{1+x^4}}\,dx = \int_0^{0.1} (1+x^4)^{-1/2}\,dx$

$= \int_0^{0.1} \left[1 + \frac{-1}{2}(x^4) + \frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)}{2!}(x^4)^2 + \frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}(x^4)^3 + \cdots\right] dx$

$= \int_0^{0.1} \left[1 - \frac{1}{2}x^4 + \frac{3}{8}x^8 - \frac{5}{16}x^{12} + \cdots\right] dx$

$= \left(x - \frac{1}{10}x^5 + \frac{3}{72}x^9 - \cdots\right)\Big|_0^{0.1}$

$= (0.1) - \frac{1}{10}(0.1)^5 + \frac{3}{72}(0.1)^9 - \cdots$

$(0.1)^6 = 0.000001 \leq 0.001 = 10^{-3}$

so by convergent alternating series facts

$\int_0^{0.1} \frac{1}{\sqrt{1+x^4}}\,dx \approx 0.1$ with absolute error

$|R_1| \leq (0.1)^6 = 0.000001 \leq 10^{-3}$

20.) $\int_0^{0.1} e^{-x^2}\,dx = \int_0^{0.1} \left(1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots\right) dx$

$= \int_0^{0.1} \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right) dx$

$= \left(x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\right)\Big|_0^{0.1}$

$= (0.1) - \frac{(0.1)^3}{3} + \frac{(0.1)^5}{5\cdot 2!} - \cdots$

$\approx 0.000333 \leq 0.001 = 10^{-3}$

so by convergent alternating series facts

$\int_0^{0.1} e^{-x^2}\,dx \approx 0.1$ with absolute error

$|R_1| \leq \frac{(0.1)^3}{3} \approx 0.000333 \leq 0.001 = 10^{-3}$

22.) $\int_0^1 \frac{1-\cos x}{x^2}\,dx = \int_0^1 \frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)}{x^2}\,dx$

$= \int_0^1 \left(\frac{1}{2!}-\frac{x^2}{4!}+\frac{x^4}{6!}-\frac{x^6}{8!}+\frac{x^8}{10!}-\frac{x^{10}}{12!}+\cdots\right)$

$= \left(\frac{1}{2!}x-\frac{x^3}{4!\cdot 3}+\frac{x^5}{6!\cdot 5}-\frac{x^7}{8!\cdot 7}+\frac{x^9}{10!\cdot 9}-\frac{x^{11}}{12!\cdot 11}+\cdots\right)\Big|_0^1$

$= \frac{1}{2!}-\frac{1}{4!\cdot 3}+\frac{1}{6!\cdot 5}-\frac{1}{8!\cdot 7}+\frac{1}{10!\cdot 9}-\frac{1}{12!\cdot 11}+\cdots$

$\leq 1.9\times 10^{-10} < 10^{-8}$

so by convergent alternating series facts

$\int_0^1 \frac{1-\cos x}{x^2}\,dx \approx \frac{1}{2!}-\frac{1}{4!\cdot 3}+\frac{1}{6!\cdot 5}-\frac{1}{8!\cdot 7}+\frac{1}{10!\cdot 9}$

≈ 0.49288 with absolute error

$|R_5| \leq \frac{1}{12!\cdot 11} \leq 1.9\times 10^{-10} < 10^{-8}$

25.) $F(x) = \int_0^x \sin t^2 \, dt$

$$= \int_0^x \left((t^2) - \frac{(t^2)^3}{3!} + \frac{(t^2)^5}{5!} - \frac{(t^2)^7}{7!} + \cdots \right) dt$$

$$= \int_0^x \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \cdots \right) dt$$

$$= \left(\frac{t^3}{3} - \frac{t^7}{7 \cdot 3!} + \frac{t^{11}}{11 \cdot 5!} - \cdots \right) \Big|_0^x$$

$$= \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \cdots \text{ for } 0 \le x \le 1$$

$$\frac{x^{11}}{11 \cdot 5!} \le \frac{1^{11}}{11 \cdot 5!} \approx 0.00075 \le 10^{-3}$$

so by convergent alternating series facts

$F(x) \approx \frac{x^3}{3} - \frac{x^7}{42}$ with absolute error

$$|R_2| \le \frac{x^{11}}{11 \cdot 5!} \le 0.00075 \le 10^{-3}.$$

28.) $F(x) = \int_0^x \frac{\ln(1+t)}{t} \, dt$

$$= \int_0^x \frac{1}{t} \cdot \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \right) dt$$

$$= \int_0^x \left(1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \cdots \right) dt$$

$$= \left(t - \frac{t^2}{4} + \frac{t^3}{9} - \frac{t^4}{16} \right) \Big|_0^x$$

$$= x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n^2} + \cdots$$

a.) If $0 \le x \le 0.5$, then

$\frac{x^6}{6^2} \le \frac{(0.5)^6}{6^2} \approx 0.000434 \le 10^{-3}$, so

by convergent alternating series facts

$$F(x) \approx x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \frac{x^5}{25}$$

with absolute error

$$|R_5| \le \frac{x^6}{36} \le \frac{(0.5)^6}{36} \approx 0.000434 \le 10^{-3}.$$

b.) If $0 \le x \le 1$, then

$$\frac{x^{32}}{32^2} \le \frac{1^{32}}{1024} = \frac{1}{1024} \le \frac{1}{1000} = 10^{-3},$$ so

by convergent alternating series facts

$$F(x) \approx x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \cdots + \frac{x^{31}}{31^2} - \frac{x^{32}}{32^2}$$

with absolute error

$$|R_{32}| \le \frac{x^{32}}{32^2} \le \frac{1^{32}}{32^2} = \frac{1}{1024} \le \frac{1}{1000} = 10^{-3}$$

29.) a.) $$\lim_{x \to 0} \frac{e^x - (1+x)}{x^2} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to 0} \frac{e^x - 1}{2x}$$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$$

b.) $$\lim_{x \to 0} \frac{e^x - (1+x)}{x^2}$$

$$= \lim_{x \to 0} \frac{\left(\cancel{1+x} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \left(\cancel{1+x}\right)}{x^2}$$

$$= \lim_{x \to 0} \frac{\cancel{x^2}\left(\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \cdots\right)}{\cancel{x^2}}$$

$$= \frac{1}{2} + 0 + 0 + 0 + \cdots = 1/2$$

32.) a.) $\lim_{\theta \to 0} \dfrac{\sin\theta - \theta + \frac{\theta^3}{6}}{\theta^5}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{\theta \to 0} \dfrac{\cos\theta - 1 + \frac{\theta^2}{2}}{5\theta^4} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{\theta \to 0} \dfrac{-\sin\theta + \theta}{20\theta^3}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{\theta \to 0} \dfrac{-\cos\theta + 1}{60\theta^2} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{\theta \to 0} \dfrac{\sin\theta}{120\theta}$

$= \dfrac{1}{120} \cdot \lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = \dfrac{1}{120} \cdot 1 = \dfrac{1}{120}$

b.) $\lim_{\theta \to 0} \dfrac{\sin\theta - \theta + \theta^3/6}{\theta^5}$

$= \lim_{\theta \to 0} \dfrac{\left(\cancel{\theta} - \cancel{\frac{\theta^3}{3!}} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) - \cancel{\theta} + \cancel{\frac{\theta^3}{3!}}}{\theta^5}$

$= \lim_{\theta \to 0} \dfrac{\cancel{\theta^5} \cdot \left(\frac{1}{5!} - \frac{\theta^2}{7!} + \frac{\theta^4}{9!} - \cdots\right)}{\cancel{\theta^5}}$

$= \dfrac{1}{5!} - 0 + 0 - 0 + 0 - \cdots = \dfrac{1}{120}$

34.) a.) $\lim_{Y \to 0} \dfrac{\arctan Y - \sin Y}{Y^3 \cos Y}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{Y \to 0} \dfrac{\frac{1}{1+Y^2} - \cos Y}{-Y^3 \sin Y + 3Y^2 \cos Y}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{Y \to 0} \dfrac{\frac{-2Y}{(1+Y^2)^2} + \sin Y}{-Y^3 \cos Y - 3Y^2 \sin Y - 3Y^2 \sin Y + 6Y \cos Y}$

$$= \lim_{Y \to 0} \frac{\cancel{X} \cdot \left[\frac{-2}{(1+Y^2)^2} + \frac{\sin Y}{Y} \right]}{\cancel{X} \cdot \left[-Y^2 \cos Y - 3Y \sin Y - 3Y \sin Y + 6 \cos Y \right]}$$

$$= \frac{\frac{-2}{1^2} + 1}{-0-0-0+6\cos 0} = \frac{-1}{6}$$

6.) $\lim_{Y \to 0} \frac{\arctan Y - \sin Y}{y^3 \cos Y}$

$$= \lim_{Y \to 0} \frac{\left(\cancel{Y} - \frac{Y^3}{3} + \frac{Y^5}{5} - \frac{Y^7}{7} + \cdots\right) - \left(\cancel{Y} - \frac{Y^3}{3!} + \frac{Y^5}{5!} - \frac{Y^7}{7!} + \cdots\right)}{Y^3 \left(1 - \frac{Y^2}{2!} + \frac{Y^4}{4!} - \frac{Y^6}{6!} + \cdots\right)}$$

$$= \lim_{Y \to 0} \frac{\frac{-1}{6} Y^3 + \frac{23}{120} Y^5 - \frac{719}{5040} Y^7 + \cdots}{Y^3 \left(1 - \frac{Y^2}{2!} + \frac{Y^4}{4!} - \frac{Y^6}{6!} + \cdots\right)}$$

$$= \lim_{Y \to 0} \frac{\cancel{Y^3} \cdot \left(\frac{-1}{6} + \frac{23}{120} Y^2 - \frac{719}{5040} Y^4 + \cdots\right)}{\cancel{Y^3} \cdot \left(1 - \frac{Y^2}{2!} + \frac{Y^4}{4!} - \frac{Y^6}{6!} + \cdots\right)}$$

$$= \frac{-1/6 + 0 - 0 + 0 - 0 + \cdots}{1 - 0 + 0 - 0 + 0 - \cdots} = -1/6$$

38.) a.) $\lim_{X \to 2} \frac{X^2 - 4}{\ln(X-1)} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{X \to 2} \frac{2X}{\frac{1}{X-1}}$

$$= \frac{4}{\frac{1}{1}} = 4$$

b.) $$\lim_{x\to 2} \frac{x^2-4}{\ln(x-1)} = \lim_{x\to 2} \frac{(x-2)(x+2)}{\ln(1+(x-2))}$$

$$= \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2) - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} - \frac{(x-2)^4}{4} + \cdots}$$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}\left[1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{3} - \frac{(x-2)^3}{4} + \cdots\right.}$$

$$= \frac{2+2}{1-0+0-0+\cdots}$$

$$= 4$$

39.) a.) $$\lim_{x\to 0} \frac{\sin(3x^2)}{1-\cos 2x}$$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0} \frac{3\cancel{6}x\cdot\cos(3x^2)}{\cancel{2}\sin(2x)}$$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0} \frac{3x\cdot{}^{-}6x\cdot\sin(3x^2) + 3\cdot\cos(3x^2)}{2\cos 2x}$$

$$= \frac{0 + 3\cos 0}{2\cdot\cos 0}$$

$$= \frac{3(1)}{2(1)}$$

$$= \frac{3}{2}$$

b.) $\lim_{x\to 0} \dfrac{\sin(3x^2)}{1-\cos 2x}$

$$= \lim_{x\to 0} \frac{(3x^2) - \frac{(3x^2)^3}{3!} + \frac{(3x^2)^5}{5!} - \cdots}{1 - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right)}$$

$$= \lim_{x\to 0} \frac{3x^2 - \frac{3^3}{3!}x^6 + \frac{3^5 x^{10}}{5!} - \cdots}{\frac{2^2}{2!}x^2 - \frac{2^4 x^4}{4!} + \frac{2^6 x^6}{6!} - \cdots}$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}\left(3 - \frac{3^3}{3!}x^4 + \frac{3^5}{5!}x^8 - \cdots\right)}{\cancel{x^2}\left(2 - \frac{2^4}{4!}x^2 + \frac{2^6}{6!}x^4 - \cdots\right)}$$

$$= \frac{3 - 0 + 0 - 0 + \cdots}{2 - 0 + 0 - \cdots} = \frac{3}{2}$$

41.) $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e^1$

42.) $\left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5 + \left(\frac{1}{4}\right)^6 + \cdots$

$$= \left(\frac{1}{4}\right)^3 \cdot \left[1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots\right]$$

$$= \frac{1}{64} \cdot \frac{1}{1-\frac{1}{4}} = \frac{1}{64} \cdot \frac{4}{3} = \frac{1}{48}$$

43.) $1 - \frac{3^2}{4^2 2!} + \frac{3^4}{4^4 4!} - \frac{3^6}{4^6 6!} + \cdots$

$$= 1 - \frac{\left(\frac{3}{4}\right)^2}{2!} + \frac{\left(\frac{3}{4}\right)^4}{4!} - \frac{\left(\frac{3}{4}\right)^6}{6!} + \cdots = \cos\left(\frac{3}{4}\right)$$

44.) $\frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \cdots$

$$= \left(\frac{1}{2}\right) - \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} - \frac{\left(\frac{1}{2}\right)^4}{4} + \cdots$$

$$= \ln\left(1 + \left(\frac{1}{2}\right)\right) = \ln\left(\frac{3}{2}\right)$$

45.) $\frac{\pi}{3} - \frac{\pi^3}{3^3 \cdot 3!} + \frac{\pi^5}{3^5 \cdot 5!} - \frac{\pi^7}{3^7 \cdot 7!} + \cdots$

$$= \left(\frac{\pi}{3}\right) - \frac{\left(\frac{\pi}{3}\right)^3}{3!} + \frac{\left(\frac{\pi}{3}\right)^5}{5!} - \frac{\left(\frac{\pi}{3}\right)^7}{7!} + \cdots$$

$$= \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

46.) $\frac{2}{3} - \frac{2^3}{3^3 \cdot 3} + \frac{2^5}{3^5 \cdot 5} - \frac{2^7}{3^7 \cdot 7} + \cdots$

$$= \left(\frac{2}{3}\right) - \frac{\left(\frac{2}{3}\right)^3}{3} + \frac{\left(\frac{2}{3}\right)^5}{5} - \frac{\left(\frac{2}{3}\right)^7}{7} + \cdots = \arctan\left(\frac{2}{3}\right)$$

47.) $x^3 + x^4 + x^5 + x^6 + \cdots$

$$= x^3\left(1 + x + x^2 + x^3 + \cdots\right)$$

$$= x^3 \cdot \frac{1}{1-x} = \frac{x^3}{1-x}$$

48.) $1 - \frac{3^2 x^2}{2!} + \frac{3^4 x^4}{4!} - \frac{3^6 x^6}{6!} + \cdots$

$$= 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \cdots$$

$$= \cos(3x)$$

49.) $x^3 - x^5 + x^7 - x^9 + x^{11} - \cdots$

$$= x^3(1 - x^2 + x^4 - x^6 + \cdots)$$

$$= x^3(1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots)$$

$$= x^3 \cdot \frac{1}{1-(-x^2)} = \frac{x^3}{1+x^2}$$

50.) $x^2 - 2x^3 + \frac{2^2x^4}{2!} - \frac{2^3x^5}{3!} + \frac{2^4x^6}{4!} - \cdots$

$$= x^2 - (2x)x^2 + \frac{(2x)^2x^2}{2!} - \frac{(2x)^3x^2}{3!} + \frac{(2x)^4x^2}{4!} - \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \xrightarrow[\text{FOR } x]{\text{SUB } -2x}$$

$$e^{-2x} = 1 - 2x + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} - \cdots \xrightarrow[x^2]{\text{mult.}}$$

$$x^2e^{-2x} = x^2 - 2x^3 + \frac{2^2x^4}{2!} - \frac{2^3x^5}{3!} + \frac{2^4x^6}{4!} - \cdots$$

51.) $-1+2x-3x^2+4x^3-5x^4+\cdots$

$= D\left(-x+x^2-x^3+x^4-x^5+\cdots\right)$

$= D\left[-x\cdot\left(1-x+x^2-x^3+x^4-\cdots\right)\right]$

$= D\left[-x\cdot\frac{1}{1+x}\right] = D\left(\frac{-x}{1+x}\right)$

$= \frac{(1+x)(-1)-(-x)(1)}{(1+x)^2} = \frac{-1-x+x}{(1+x)^2}$

$= \frac{-1}{(1+x)^2}$.

52.) $1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\frac{x^4}{5}+\cdots$

$$= \frac{1}{x} \cdot x\left(1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots\right)$$

$$= \frac{1}{x} \cdot \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$$

$$= \frac{1}{x} \cdot \int_0^x (1 + t + t^2 + t^3 + \cdots)\, dt$$

$$= \frac{1}{x} \cdot \int_0^x \frac{1}{1-t}\, dt$$

$$= \frac{1}{x} \cdot -\ln(1-t)\Big|_0^x$$

$$= \frac{1}{x} \cdot \left(-\ln(1-x) - -\cancelto{0}{\ln(1)}\right)$$

$$= \frac{-\ln(1-x)}{x}$$

53.) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \longrightarrow$

$$\ln(1-x) = \ln(1+(-x)) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \cdots$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \text{, then}$$

$$\ln \frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right)$$

$$= 2x + 2 \cdot \frac{x^3}{3} + 2 \cdot \frac{x^5}{5} + \cdots$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

61.) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots \longrightarrow$

$$\frac{-1}{1+x} = -1 + x - x^2 + x^3 - x^4 + \cdots \xrightarrow{D}$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

62.) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \longrightarrow$

$$\frac{1}{1-x^2} = 1 + (x^2) + (x^2)^2 + (x^2)^3 + (x^2)^4 + \cdots$$

$$= 1 + x^2 + x^4 + x^6 + x^8 + \cdots \xrightarrow{D}$$

$$\frac{2x}{(1-x^2)^2} = 2x + 4x^3 + 6x^5 + 8x^7 + \cdots$$

$$= \sum_{n=0}^{\infty} 2(n+1) x^{2n+1}$$

65.) $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \longrightarrow$

$$\arcsin x = \int \frac{1}{\sqrt{1-x^2}}\,dx = \int \left(1+(-x^2)\right)^{-1/2} dx$$

$$= \int \left[1 + \frac{-1}{2}(-x^2) + \frac{-\frac{1}{2}\left(\frac{-1}{2}-1\right)}{2!}(-x^2)^2 + \frac{-\frac{1}{2}\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}(-x^2)^3 + \cdots\right] dx$$

$$= \int \left[1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{15}{48}x^6 + \cdots\right] dx$$

$$= x + \frac{1}{2}\,\frac{x^3}{3} + \frac{3}{8}\,\frac{x^5}{5} + \frac{15}{48}\cdot\frac{x^7}{7} + \cdots$$

$$= x + \frac{1}{(2)\,3}x^3 + \frac{3}{(2\cdot 4)\,5}x^5 + \frac{3\cdot 5}{(2\cdot 4\cdot 6)\cdot 7}x^7 + \cdots$$

$$= x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1}$$