Section 12.4

2.) $\vec{u} = \overrightarrow{(2,3,0)}$, $\vec{v} = \overrightarrow{(-1,1,0)}$ then

a.) $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 0 \\ -1 & 1 & 0 \end{vmatrix}$

$= \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \vec{k}$

$= (0)\vec{i} - (0)\vec{j} + 5\vec{k} \quad = 5\vec{k}$;

$|\vec{u} \times \vec{v}| = 5$ and direction of $\vec{u} \times \vec{v}$ is $\vec{k}$.

b.) $\vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ 2 & 3 & 0 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix} \vec{k}$

$= (0)\vec{i} - (0)\vec{j} + (-5)\vec{k} \quad = -5\vec{k}$;

$|\vec{v} \times \vec{u}| = 5$ and direction of $\vec{v} \times \vec{u}$ is $-\vec{k}$

3.) $\vec{u} = (2,-2,4)$, $\vec{v} = (-1,1,-2)$ then

a.) $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix}$

$= \begin{vmatrix} -2 & 4 \\ 1 & -2 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 4 \\ -1 & -2 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} \vec{k}$

$= (0)\vec{i} - (0)\vec{j} + (0)\vec{k} \quad = \overrightarrow{(0,0,0)}$;

$|\vec{u} \times \vec{v}| = 0$ and $\vec{u} \times \vec{v}$ has no direction

b.) $\vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{vmatrix}$

$$= \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} \vec{i} - \begin{vmatrix} -1 & -2 \\ 2 & 4 \end{vmatrix} \vec{j} + \begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} \vec{k}$$

$$= (0)\vec{i} - (0)\vec{j} + (0)\vec{k} = \overrightarrow{(0,0,0)} \ ;$$

$|\vec{v} \times \vec{u}| = 0$ and $\vec{v} \times \vec{u}$ has no direction.

7.) $\vec{u} = \overrightarrow{(-8,-2,-4)}$ $\vec{v} = \overrightarrow{(2,2,1)}$ then

a.) $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -8 & -2 & -4 \\ 2 & 2 & 1 \end{vmatrix}$

$$= \begin{vmatrix} -2 & -4 \\ 2 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} -8 & -4 \\ 2 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} -8 & -2 \\ 2 & 2 \end{vmatrix} \vec{k}$$

$$= 6\vec{i} - (0)\vec{j} + (-12)\vec{k} = \overrightarrow{(6,0,-12)} \ ;$$

$|\vec{u} \times \vec{v}| = \sqrt{36+144} = \sqrt{180} = 6\sqrt{5}$ and

direction of $\vec{u} \times \vec{v}$ is $\frac{1}{6\sqrt{5}}\overrightarrow{(6,0,-12)} = \left(\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}}\right)$

b.) $\vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 1 \\ -8 & -2 & -4 \end{vmatrix}$

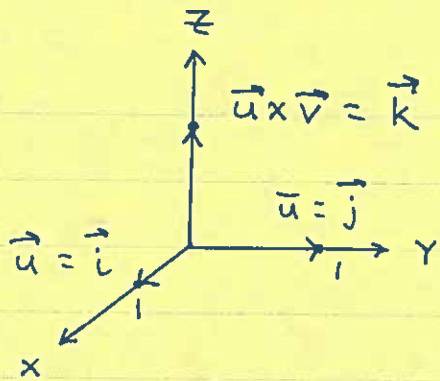
$$= \begin{vmatrix} 2 & 1 \\ -2 & -4 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 1 \\ -8 & -4 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 2 \\ -8 & -2 \end{vmatrix} \vec{k}$$

$$= (-6)\vec{i} - (0)\vec{j} + 12\vec{k} = \overrightarrow{(-6,0,12)} \ ;$$

$|\vec{v} \times \vec{u}| = \sqrt{180} = 6\sqrt{5}$ and direction

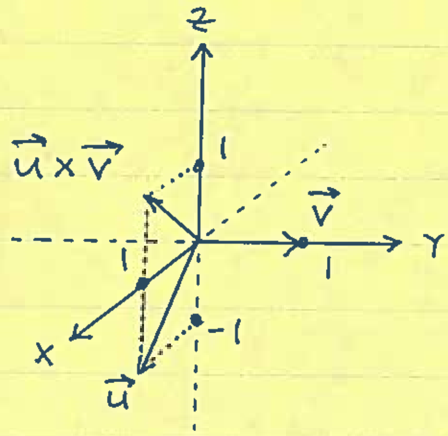
of $\vec{v} \times \vec{u}$ is $\frac{1}{6\sqrt{5}}\overrightarrow{(-6,0,12)} = \overrightarrow{\left(\frac{-1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)}$.

9.)



$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\vec{k}$$
$$= (0)\vec{i} - (0)j + (1)\vec{k} = \vec{k}$$

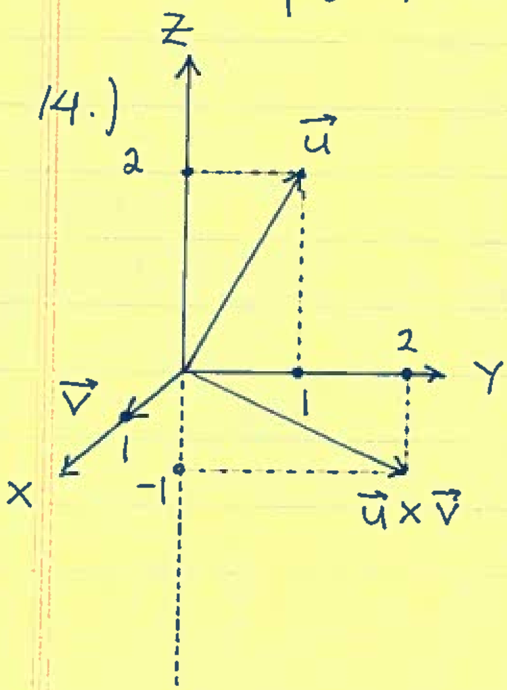
10.)



$\vec{u} = \vec{i} - \vec{k} = \overrightarrow{(1,0,-1)}$, $\vec{v} = \vec{j} = (0,1,0)$ then

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\vec{k}$$
$$= (1)\vec{i} - (0)\vec{j} + (1)\vec{k} = \vec{i} + \vec{k}$$

14.)



$\vec{u} = \vec{j} + 2\vec{k} = \overrightarrow{(0,1,2)}$,
$\vec{v} = \vec{i} = \overrightarrow{(1,0,0)}$ then

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\vec{k}$$
$$= (0)\vec{i} - (-2)\vec{j} + (-1)\vec{k}$$
$$= 2\vec{j} - \vec{k}$$

16.) $P = (1,1,1)$, $Q = (2,1,3)$, $R = (3,-1,1)$ so
$\overrightarrow{PQ} = \langle 1,0,2 \rangle$ and $\overrightarrow{PR} = \langle 2,-2,0 \rangle$;

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix} = (4)\vec{i} - (-4)\vec{j} + (-2)\vec{k}$$
$$= 4\vec{i} + 4\vec{j} - 2\vec{k} \;;$$

area of Δ is $\frac{1}{2}$ area of $\square$ so

a.) $$\text{Area} = \frac{1}{2} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right|$$
$$= \frac{1}{2}\sqrt{16+16+4} = \frac{1}{2}(6) = 3 \;;$$

b.) a unit vector $\perp$ to plane PQR is
$$\vec{u} = \frac{1}{\left| \overrightarrow{PQ} \times \overrightarrow{PR} \right|} \cdot \overrightarrow{PQ} \times \overrightarrow{PR}$$
$$= \frac{1}{6}\langle 4,4,-2 \rangle = \overrightarrow{\left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3} \right)}$$

19.) $\vec{u} = 2\vec{i}$, $\vec{v} = 2\vec{j}$, $\vec{w} = 2\vec{k}$ then

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix} = (0)\vec{i} - (0)\vec{j} + 4\vec{k} = 4\vec{k} \;,$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 4\vec{i} - (0)\vec{j} + (0)\vec{k} = 4\vec{i} \;,$$

$$\vec{w} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{vmatrix} = (0)\vec{i} - (-4)\vec{j} + (0)\vec{k}$$
$$= 4\vec{j} \;;$$

$(\vec{u} \times \vec{v}) \cdot \vec{w} = \overrightarrow{(0,0,4)} \cdot \overrightarrow{(0,0,2)} = 0+0+8 = 8,$

$(\vec{v} \times \vec{w}) \cdot \vec{u} = \overrightarrow{(4,0,0)} \cdot \overrightarrow{(2,0,0)} = 8+0+0 = 8,$

$(\vec{w} \times \vec{u}) \cdot \vec{v} = \overrightarrow{(0,4,0)} \cdot \overrightarrow{(0,2,0)} = 0+8+0 = 8;$

volume of parallelepiped is

Volume $= |(\vec{u} \times \vec{v}) \cdot \vec{w}| = |8| = 8$

20.) $\vec{u} = \overrightarrow{(1,-1,1)}$, $\vec{v} = \overrightarrow{(2,1,-2)}$, $\vec{w} = \overrightarrow{(-1,2,-1)}$ then

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = (1)\vec{i} - (-4)\vec{j} + (3)\vec{k} = \vec{i} + 4\vec{j} + 3\vec{k},$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = (3)\vec{i} - (-4)\vec{j} + (5)\vec{k} = 3\vec{i} + 4\vec{j} + 5\vec{k},$$

$$\vec{w} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} = (1)\vec{i} - (0)\vec{j} + (-1)\vec{k} = \vec{i} - \vec{k};$$

$(\vec{u} \times \vec{v}) \cdot \vec{w} = \overrightarrow{(1,4,3)} \cdot \overrightarrow{(-1,2,-1)} = -1+8-3 = 4,$

$(\vec{v} \times \vec{w}) \cdot \vec{u} = \overrightarrow{(3,4,5)} \cdot \overrightarrow{(1,-1,1)} = 3-4+5 = 4,$

$(\vec{w} \times \vec{u}) \cdot \vec{v} = \overrightarrow{(1,0,-1)} \cdot \overrightarrow{(2,1,-2)} = 2+0+2 = 4;$

volume of parallelepiped is

Volume $= |(\vec{u} \times \vec{v}) \cdot \vec{w}| = |4| = 4$

23.) $\vec{u} = 5\vec{i} - \vec{j} + \vec{k}$, $\vec{v} = \vec{j} - 5\vec{k}$,

$\vec{w} = -15\vec{i} + 3\vec{j} - 3\vec{k} = -3(5\vec{i} - \vec{j} + \vec{k}) = -3\vec{u}$:

a.) $\vec{u}\cdot\vec{v} = (5)(0)+(-1)(1)+(1)(-5) = -6$,

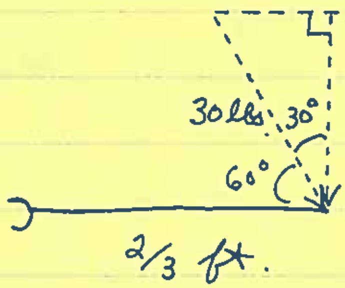
$\vec{v}\cdot\vec{w} = \vec{v}\cdot(-3\vec{u}) = -3(\vec{u}\cdot\vec{v}) = -3(-6) = 18$,

$\vec{u}\cdot\vec{w} = \vec{u}\cdot(-3\vec{u}) = -3\left((5)^2+(-1)^2+(1)^2\right) = -81$;

so no vectors are $\perp$

b.) $\vec{w} = -3\vec{u}$ so $\vec{u}$ and $\vec{w}$ are $\parallel$.

25.)



$\frac{\sqrt{3}}{2}\cdot(30) = 15\sqrt{3}$ lbs.

Torque = (vert. force)(distance)
$= (15\sqrt{3})(2/3) = 10\sqrt{3}$ ft.-lbs.

27.)
a.) True
b.) False
c.) True
d.) True
e.) False
f.) True
g.) True
h.) True

28.)
a.) True
b.) True
c.) True
d.) True
e.) True
f.) True
g.) True
h.) True

29.) a.) $\text{proj.}_{\vec{v}}\,\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$

b.) $\vec{u}\times\vec{v}$ is $\perp$ to $\vec{u}$ and $\vec{v}$

c.) $(\vec{u}\times\vec{v})\times\vec{w}$ is $\perp$ to $\vec{u}\times\vec{v}$ and $\vec{w}$

d.) Volume of parallelepiped is
$|(\vec{u}\times\vec{v})\cdot\vec{w}|$

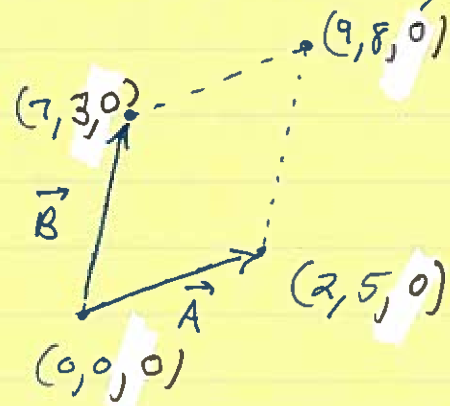
e.) $(\vec{u}\times\vec{v})\times(\vec{u}\times\vec{w})$ is $\perp$ to $\vec{u}\times\vec{v}$ and $\vec{u}\times\vec{w}$

f.) unit vector in direction of $\vec{v}$ is $\dfrac{\vec{v}}{|\vec{v}|}$, so $|\vec{u}|\dfrac{\vec{v}}{|\vec{v}|}$ has length $|\vec{u}|$

31.) a.) Sense
b.) Not
c.) Sense
d.) Not

33.) Not necessarily: Let $\vec{v}=\vec{0}$ and $\vec{w}=2\vec{u}$. Then
$\vec{u}\times\vec{v}=\vec{u}\times\vec{0}=\vec{0}$ and
$\vec{u}\times\vec{w}=\vec{u}\times(2\vec{u})=2(\vec{u}\times\vec{u})=2(\vec{0})=\vec{0}$,
but $\vec{v}\neq\vec{w}$.

36.) $\vec{A}=\overrightarrow{(2,5,0)}$, $\vec{B}=\overrightarrow{(7,3,0)}$



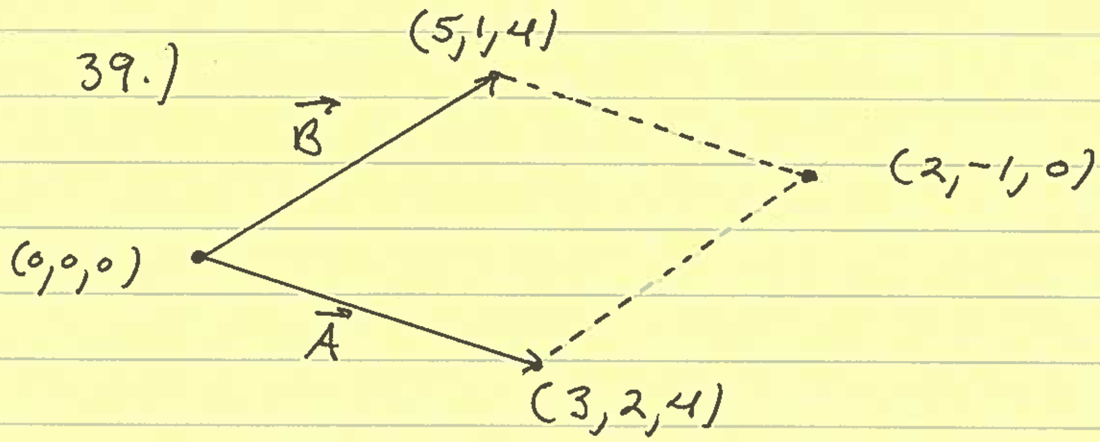
so area of parallelogram is $|\vec{A}\times\vec{B}|$:

$$\vec{A}\times\vec{B}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\2&5&0\\7&3&0\end{vmatrix}=\begin{vmatrix}5&0\\3&0\end{vmatrix}\vec{i}-\begin{vmatrix}2&0\\7&0\end{vmatrix}\vec{j}+\begin{vmatrix}2&5\\7&3\end{vmatrix}\vec{k}$$

$$=(0-0)\vec{i}-(0-0)\vec{j}+(-29)\vec{k}=-29\vec{k},$$

so Area $=|\vec{A}\times\vec{B}|=|-29\vec{k}|=\boxed{29}$

39.)



$\vec{A} = \overrightarrow{(3,2,4)}$, $\vec{B} = \overrightarrow{(5,1,4)}$ so area of parallelogram is $|\vec{A} \times \vec{B}|$:

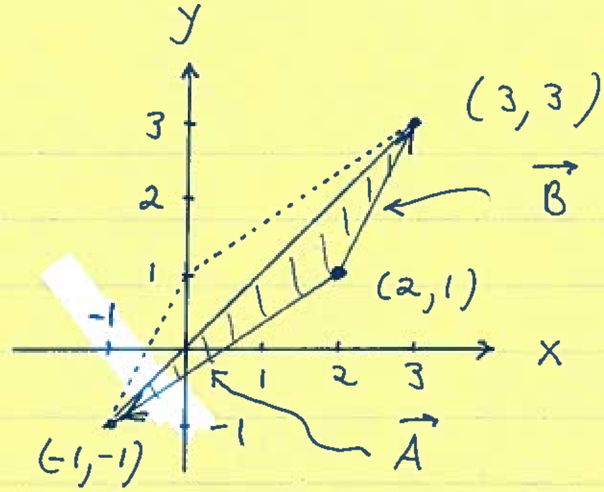
$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 4 \\ 5 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} \vec{i} - \begin{vmatrix} 3 & 4 \\ 5 & 4 \end{vmatrix} \vec{j} + \begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix} \vec{k}$$

$$= (8-4)\vec{i} - (12-20)\vec{j} + (3-10)\vec{k}$$

$$= 4\vec{i} + 8\vec{j} - 7\vec{k}$$, so

$$\text{Area} = |\vec{A} \times \vec{B}| = \sqrt{(4)^2 + (8)^2 + (-7)^2} = \boxed{\sqrt{129}}$$

42)



Area of triangle is $\frac{1}{2}$ area of parallelogram ; so

$\vec{A} = \overrightarrow{(-3,-2,0)}$, $\vec{B} = \overrightarrow{(1,2,0)}$,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & -2 & 0 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 0 \\ 2 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} -3 & 0 \\ 1 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} -3 & -2 \\ 1 & 2 \end{vmatrix} \vec{k}$$

$$= (0-0)\,\vec{i} - (0-0)\,\vec{j} + (-4)\,\vec{k} = -4\vec{k} \;;$$

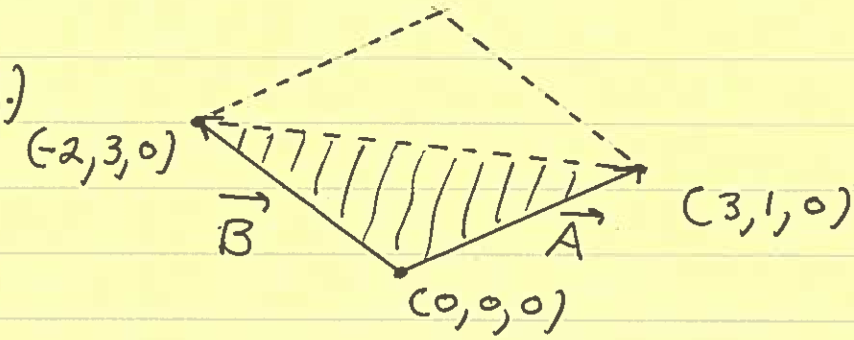
$$\text{Area of } \Delta = \frac{1}{2} \left| \vec{A} \times \vec{B} \right|$$

$$= \frac{1}{2} \left| -4\vec{k} \right|$$

$$= \frac{1}{2}(4)$$

$$= \boxed{2} \; .$$

46.)



Note that area of triangle is half the area of the parallelogram.

$\vec{A} = \overrightarrow{(3,1,0)}$, $\vec{B} = \overrightarrow{(-2,3,0)}$

so area of parallelogram is $|\vec{A}\times\vec{B}|$:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 0 \\ -2 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} 3 & 0 \\ -2 & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix}\vec{k}$$

$$= (0-0)\vec{i} - (0-0)\vec{j} + (9-(-2))\vec{k} = -11\,\vec{k},$$

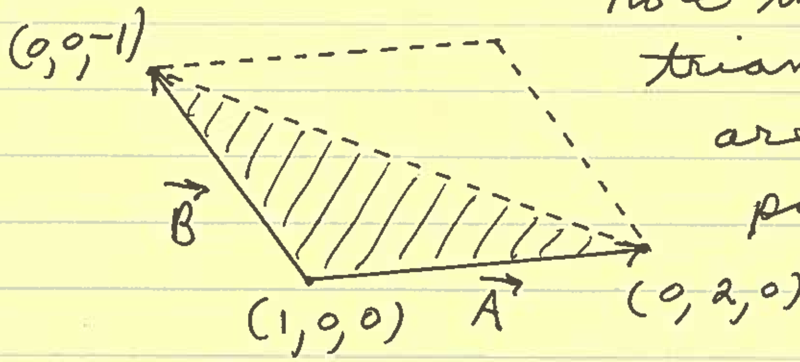
so Area of triangle

$$= \frac{1}{2}(\text{Area of parallelogram})$$

$$= \frac{1}{2}|\vec{A}\times\vec{B}| = \frac{1}{2}(11) = \boxed{5.5}$$

45.)

Note that area of triangle half the area of the parallelogram.



$\vec{A} = \overrightarrow{\langle 0-1, 2-0, 0-0 \rangle} = \overrightarrow{\langle -1, 2, 0 \rangle}$,

$\vec{B} = \overrightarrow{\langle 0-1, 0-0, -1-0 \rangle} = \overrightarrow{\langle -1, 0, -1 \rangle}$ so area of parallelogram is $|\vec{A} \times \vec{B}|$:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ -1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} -1 & 0 \\ -1 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix} \vec{k}$$

$$= (-2)\vec{i} - (1-0)\vec{j} + (0-(-2))\vec{k}$$

$$= -2\vec{i} - \vec{j} + 2\vec{k}$$, so

$$\text{Area of triangle} = \frac{1}{2}(\text{Area of parallelogram})$$

$$= \frac{1}{2}|\vec{A} \times \vec{B}|$$

$$= \frac{1}{2}\sqrt{(-2)^2 + (-1)^2 + (2)^2}$$

$$= \frac{1}{2}\sqrt{9} = \boxed{\frac{3}{2}}$$

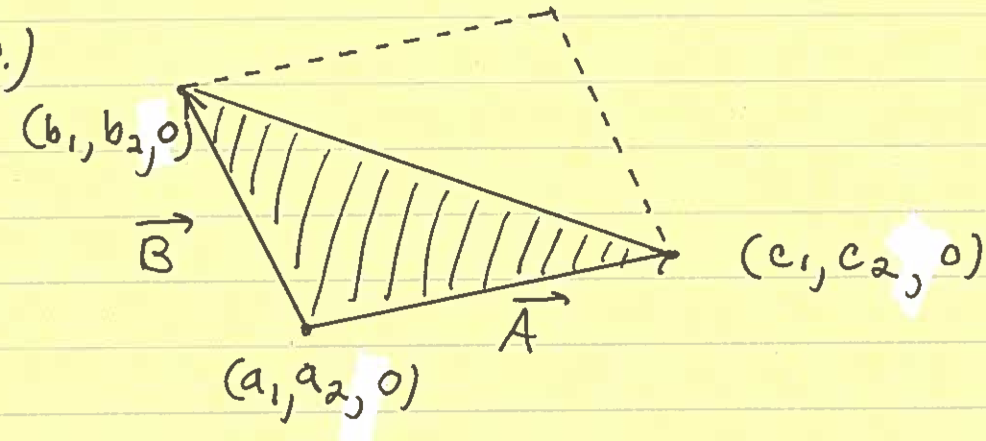
48.) $\vec{AB} = \overrightarrow{\langle 1, 2, 0 \rangle}$, $\vec{AC} = \overrightarrow{\langle 0, -3, 2 \rangle}$, $\vec{AD} = \overrightarrow{\langle 3, -4, 5 \rangle}$, so volume of parallelepiped is

$$\begin{vmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 3 & -4 & 5 \end{vmatrix} = (1)\begin{vmatrix} -3 & 2 \\ -4 & 5 \end{vmatrix} - (2)\begin{vmatrix} 0 & 2 \\ 3 & 5 \end{vmatrix} - (0)\begin{vmatrix} 0 & -3 \\ 3 & -4 \end{vmatrix}$$

$$= (-15-(-8)) - 2(0-6) - 0$$

$$= -7 + 12 = \boxed{5}$$

50.)



Note that area of triangle is half the area of the parallelogram.

$\vec{A} = \overrightarrow{(c_1 - a_1, c_2 - a_2, 0)}$, $\vec{B} = \overrightarrow{(b_1 - a_1, b_2 - a_2, 0)}$

so area of parallelogram is $|\vec{A} \times \vec{B}|$:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ c_1 - a_1 & c_2 - a_2 & 0 \\ b_1 - a_1 & b_2 - a_2 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} c_2 - a_2 & 0 \\ b_2 - a_2 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} c_1 - a_1 & 0 \\ b_1 - a_1 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} c_1 - a_1 & c_2 - a_2 \\ b_1 - a_1 & b_2 - a_2 \end{vmatrix} \vec{k}$$

$$= (0-0)\vec{i} - (0-0)\vec{j} + \{(c_1 - a_1)(b_2 - a_2) - (c_2 - a_2)(b_1 - a_1)\}\vec{k},$$

so $|\vec{A} \times \vec{B}| = |(c_1 - a_1)(b_2 - a_2) - (c_2 - a_2)(b_1 - a_1)|$

and area of triangle is

$$\boxed{\text{Area} = \frac{1}{2}|(c_1 - a_1)(b_2 - a_2) - (c_2 - a_2)(b_1 - a_1)|}$$