Section 14.3

2.) $f(x,y) = x^2 - xy + y^2 \longrightarrow$

$$\frac{\partial f}{\partial x} = 2x - y \quad , \quad \frac{\partial f}{\partial y} = -x + 2y$$

4.) $f(x,y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2 \longrightarrow$

$$\frac{\partial f}{\partial x} = 5y - 14x + 3 \quad , \quad \frac{\partial f}{\partial y} = 5x - 2y - 6$$

5.) $f(x,y) = (xy-1)^2 \longrightarrow$

$$\frac{\partial f}{\partial x} = 2(xy-1)\cdot y \quad , \quad \frac{\partial f}{\partial y} = 2(xy-1)\cdot x$$

7.) $f(x,y) = (x^2+y^2)^{1/2} \longrightarrow$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2}} \quad ,$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2+y^2}}$$

10.) $f(x,y) = \dfrac{x}{x^2+y^2} \longrightarrow$

$$\frac{\partial f}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad ,$$

$$\frac{\partial f}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

12.) $f(x,y) = \arctan\left(\frac{y}{x}\right) \longrightarrow$

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{-y}{x^2} \quad , \quad \frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}$$

13.) $f(x,y) = e^{x+y+1} \longrightarrow$

$\frac{\partial f}{\partial x} = e^{x+y+1} \cdot (1)$, $\frac{\partial f}{\partial y} = e^{x+y+1} \cdot (1)$

15.) $f(x,y) = \ln(3x+y^2) \longrightarrow$

$\frac{\partial f}{\partial x} = \frac{1}{3x+y^2} \cdot (3)$, $\frac{\partial f}{\partial y} = \frac{1}{3x+y^2} \cdot (2y)$

16.) $f(x,y) = e^{xy} \ln y \longrightarrow$

$\frac{\partial f}{\partial x} = e^{xy} \cdot (y) \cdot \ln y + e^{xy} \cdot (0) = ye^{xy} \ln y$,

$\frac{\partial f}{\partial y} = e^{xy} \cdot \frac{1}{y} + xe^{xy} \cdot \ln y = e^{xy}\left(\frac{1}{y} + x \ln y\right)$

19.) $f(x,y) = x^y \longrightarrow \frac{\partial f}{\partial x} = yx^{y-1}$ and

$\frac{\partial f}{\partial y} = x^y \cdot \ln x$

20.) $f(x,y) = \log_y x = \frac{\ln x}{\ln y} \longrightarrow$

$\frac{\partial f}{\partial x} = \frac{1}{\ln y} \cdot \frac{1}{x}$, $\frac{\partial f}{\partial y} = \ln x \cdot -(\ln y)^{-2} \cdot \frac{1}{y}$

21.) $f(x,y) = \int_x^y g(t)\,dt \rightarrow f(x,y) = -\int_y^x g(t)\,dt$;

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(-\int_y^x g(t)\,dt\right) = -g(x)$ (by FTC 1),

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int_x^y g(t)\,dt\right) = g(y)$ (by FTC 1)

41.) $f(x,y) = x+y+xy \rightarrow$

$f_x = 1+y$, $f_y = 1+x$, $f_{xx} = 0$, $f_{xy} = 1$,

$f_{yx} = 1$, $f_{yy} = 0$

42.) $f(x,y) = \sin xy \rightarrow$

$f_x = \cos xy \cdot y$, $f_y = \cos xy \cdot x$,

$f_{xx} = y \cdot {}^{-}\sin xy \cdot y = -y^2 \sin xy$,

$f_{xy} = \cos xy \cdot (1) + {}^{-}\sin xy \cdot x \cdot y$

$= \cos xy - xy \sin xy$,

$f_{yx} = \cos xy \cdot (1) + {}^{-}\sin xy \cdot y \cdot x$

$= \cos xy - xy \sin xy$

$f_{yy} = x \cdot {}^{-}\sin xy \cdot x = -x^2 \sin xy$

43.) $g(x,y) = x^2 y + \cos y + y \sin x \rightarrow$

$g_x = 2xy + y\cos x$, $g_y = x^2 - \sin y + \sin x$,

$g_{xx} = 2y - y\sin x$, $g_{yy} = -\cos y$,

$g_{xy} = 2x + \cos x$, $g_{yx} = 2x + \cos x$

44.) $h(x,y) = xe^y + y + 1 \longrightarrow$

$h_x = e^y$, $h_y = xe^y + 1$, $h_{xy} = e^y$, $h_{yx} = e^y$,

$h_{xx} = 0$, $h_{yy} = xe^y$

45.) $r(x,y) = \ln(x+y) \longrightarrow$

$r_x = \dfrac{1}{x+y}$, $r_y = \dfrac{1}{x+y}$,

$r_{xx} = \dfrac{-1}{(x+y)^2}$, $r_{yy} = \dfrac{-1}{(x+y)^2}$,

$r_{xy} = \dfrac{-1}{(x+y)^2}$, $r_{yx} = \dfrac{-1}{(x+y)^2}$

46.) $s(x,y) = \arctan\left(\dfrac{y}{x}\right) \longrightarrow$

$s_x = \dfrac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \dfrac{-y}{x^2} = \dfrac{-y}{x^2+y^2}$,

$s_y = \dfrac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \dfrac{1}{x} = \dfrac{1}{x+\frac{y^2}{x}} \cdot \dfrac{x}{x} = \dfrac{x}{x^2+y^2}$,

$s_{xx} = \dfrac{(x^2+y^2)(0) - (-y)(2x)}{(x^2+y^2)^2} = \dfrac{2xy}{(x^2+y^2)^2}$,

$s_{yy} = \dfrac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = \dfrac{-2xy}{(x^2+y^2)^2}$,

$s_{xy} = \dfrac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$,

$s_{yx} = \dfrac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

48.) $w = ye^{x^2-y} \rightarrow$

$w_x = y \cdot e^{x^2-y} \cdot 2x = 2xye^{x^2-y}$,

$w_y = y \cdot e^{x^2-y} \cdot (-1) + (1)e^{x^2-y} = e^{x^2-y} - ye^{x^2-y}$,

$w_{xx} = 2xy \cdot e^{x^2-y} \cdot (2x) + 2y \cdot e^{x^2-y}$
$= 4x^2ye^{x^2-y} + 2ye^{x^2-y}$,

$w_{yy} = e^{x^2-y} (-1) - [ye^{x^2-y} (-1) + (1)e^{x^2-y}]$
$= -e^{x^2-y} + ye^{x^2-y} - e^{x^2-y}$
$= ye^{x^2-y} - 2e^{x^2-y}$,

$w_{xy} = 2xy \cdot e^{x^2-y} \cdot (-1) + 2x \cdot e^{x^2-y}$
$= 2xe^{x^2-y} - 2xye^{x^2-y}$,

$w_{yx} = e^{x^2-y} (2x) - [ye^{x^2-y} \cdot (2x) + (0) \cdot e^{x^2-y}]$
$= 2xe^{x^2-y} - 2xye^{x^2-y}$

51.) $w = \ln(2x+3y) \rightarrow$

$w_x = \frac{2}{2x+3y}$, $w_y = \frac{3}{2x+3y}$,

$w_{xy} = 2 \cdot -(2x+3y)^{-2} \cdot (3) = \frac{-6}{(2x+3y)^2}$,

$w_{yx} = 3 \cdot -(2x+3y)^{-2} \cdot (2) = \frac{-6}{(2x+3y)^2}$

52.) $w = e^x + x \ln y + y \ln x \longrightarrow$

$w_x = e^x + \ln y + y \cdot \frac{1}{x}$,

$w_y = x \cdot \frac{1}{y} + \ln x$,

$w_{xy} = \frac{1}{y} + \frac{1}{x}$, $w_{yx} = \frac{1}{y} + \frac{1}{x}$

53.) $w = xy^2 + x^2y^3 + x^3y^4 \longrightarrow$

$w_x = y^2 + 2xy^3 + 3x^2y^4$,

$w_y = 2xy + 3x^2y^2 + 4x^3y^3$,

$w_{xy} = 2y + 6xy^2 + 12x^2y^3$,

$w_{yx} = 2y + 6xy^2 + 12x^2y^3$

54.) $w = x \sin y + y \sin x + xy \longrightarrow$

$w_x = \sin y + y \cos x + y$,

$w_y = x \cos y + \sin x + x$,

$w_{xy} = \cos y + \cos x + 1$, $w_{yx} = \cos y + \cos x + 1$

58.) $f(x,y) = 4 + 2x - 3y - xy^2$;

$$\frac{\partial f}{\partial x}(-2,1) = \lim_{h \to 0} \frac{f(-2+h, 1) - f(-2,1)}{h}$$

$$= \lim_{h \to 0} \frac{[4 + 2(-2+h) - 3(1) - (-2+h)(1)^2] - (-1)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4} - \cancel{4} + 2h - \cancel{3} + \cancel{2} - h + \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1 \ ;$$

$$\frac{\partial f}{\partial y}(-2,1) = \lim_{h\to 0} \frac{f(-2,1+h) - f(-2,1)}{h}$$

$$= \lim_{h\to 0} \frac{[4+2(-2) - 3(1+h) - (-2)(1+h)^2] - (-1)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4} - \cancel{4} - 3 - 3h + 2(1+2h+h^2) + 1}{h}$$

$$= \lim_{h\to 0} \frac{-\cancel{3} - 3h + \cancel{2} + 4h + 2h^2 + \cancel{1}}{h}$$

$$= \lim_{h\to 0} \frac{2h^2 + h}{h}$$

$$= \lim_{h\to 0} (2h+1) = 1 .$$

60.) $\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{\sin(h^3)}{h^2 \cdot h} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{h\to 0} \frac{\sin(h^3)}{h^3} = 1 ;$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(h^4)}{h^4} \cdot \frac{1}{h} = 1 \cdot \text{"}\frac{1}{0}\text{"} = \pm\infty, \text{ so D.N.E.}$$

62.) $f(x,y) = x^2 + y^3$, pt. $(-1,1)$

a.) plane $x=-1$ (y changes):

$\frac{\partial f}{\partial y} = 3y^2$ at $(-1,1)$ so

SLOPE $m = \frac{\partial f}{\partial y}(-1,1) = 3(1)^2 = 3$

b.) plane $y=1$ (x changes):

$\frac{\partial f}{\partial x} = 2x$ at $(-1,1)$ so

SLOPE $m = \frac{\partial f}{\partial x}(-1,1) = 2(-1) = -2$

65.) assume $z = f(x,y)$ and $xy + z^3x - 2yz = 0$

$\rightarrow \frac{\partial}{\partial x}\left(xy + z^3x - 2yz\right) = \frac{\partial}{\partial x}(0)$

$\rightarrow y + \left(z^3\cdot(1) + \left(3z^2\cdot z_x\right)\cdot x\right) - 2y\,z_x = 0$

$\rightarrow y + z^3 + 3xz^2\cdot z_x - 2y\,z_x = 0$

$\rightarrow (3xz^2 - 2y)\,z_x = -y - z^3$

$\rightarrow z_x = \dfrac{-y - z^3}{3xz^2 - 2y}$ (Let $x=1, y=1, z=1$)

$\rightarrow z_x = \dfrac{-1-1}{3-2} = -2$

75.) Show $f(x,y) = e^{-2y}\cos 2x$ satisfies

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad :$$

$f_x = e^{-2y}\cdot -\sin 2x\cdot 2 = -2\sin 2x\cdot e^{-2y}$,

$f_y = -2e^{-2y}\cdot\cos 2x = -2\cos 2x\cdot e^{-2y}$,

$f_{xx} = -2\cdot(2\cos 2x)\cdot e^{-2y} = -4\cos 2x\cdot e^{-2y}$,

$f_{yy} = -2\cos 2x\cdot\left(-2e^{-2y}\right) = 4\cos 2x\cdot e^{-2y}$;

then $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(-4\cos 2x\cdot e^{-2y}\right) + \left(4\cos 2x\cdot e^{-2y}\right) = 0$.

76.) Show $\ln(x^2+y^2)^{1/2} = \frac{1}{2}\ln(x^2+y^2)$

satisfies $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$:

$f_x = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x = \dfrac{x}{x^2+y^2}$,

$f_y = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2y = \dfrac{y}{x^2+y^2}$,

$$f_{xx} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2},$$

$$f_{yy} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2 - y^2}{(x^2+y^2)^2}; \text{ then}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$= \frac{\cancel{y^2} - \cancel{x^2} + \cancel{x^2} - \cancel{y^2}}{(x^2+y^2)^2} = \frac{0}{(x^2+y^2)^2} = 0.$$

77.) Show $f(x,y) = 3x + 2y - 4$ satisfies

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0:$$

$$\frac{\partial f}{\partial x} = 3 \rightarrow \frac{\partial^2 f}{\partial x^2} = 0, \quad \frac{\partial f}{\partial y} = 2 \rightarrow \frac{\partial^2 f}{\partial y^2} = 0 \text{ so}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (0) + (0) = 0$$

81.) Show $w = \sin(x+ct)$ satisfies

$$\frac{\partial^2 w}{\partial t^2} = c^2 \cdot \frac{\partial^2 w}{\partial x^2} \quad :$$

$$\frac{\partial w}{\partial t} = \cos(x+ct) \cdot c = c \cdot \cos(x+ct),$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) \cdot (1) = \cos(x+ct),$$

$$\frac{\partial^2 w}{\partial t^2} = c \cdot -\sin(x+ct) \cdot c = -c^2 \sin(x+ct),$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) \cdot (1) = -\sin(x+ct);$$

then $\dfrac{\partial^2 w}{\partial t^2} = -c^2 \sin(x+ct)$

$= c^2\left(-\sin(x+ct)\right)$

$= c^2 \cdot \dfrac{\partial^2 w}{\partial x^2}$.

84.) Show $w = \ln(2x+2ct)$ satisfies

$\dfrac{\partial^2 w}{\partial t^2} = c^2 \cdot \dfrac{\partial^2 w}{\partial x^2}$:

$\dfrac{\partial w}{\partial t} = \dfrac{1}{2x+2ct}\cdot(2c) = \dfrac{\cancel{2}c}{\cancel{2}(x+ct)} = \dfrac{c}{x+ct}$,

$\dfrac{\partial w}{\partial x} = \dfrac{1}{2x+2ct}\cdot(2) = \dfrac{\cancel{2}}{\cancel{2}(x+ct)} = \dfrac{1}{x+ct}$,

$\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial t}\left(c\cdot(x+ct)^{-1}\right) = -c(x+ct)^{-2}\cdot(c)$

$= \dfrac{-c^2}{(x+ct)^2}$,

$\dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial}{\partial x}(x+ct)^{-1} = -(x+ct)^{-2}\cdot(1) = \dfrac{-1}{(x+ct)^2}$;

then

$\dfrac{\partial^2 w}{\partial t^2} = \dfrac{-c^2}{(x+ct)^2}$

$= c^2 \cdot \dfrac{-1}{(x+ct)^2}$

$= c^2 \cdot \dfrac{\partial^2 w}{\partial x^2}$

Show

85.) $w = \tan(2x - 2ct)$ satisfies

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} :$$

$$\frac{\partial w}{\partial t} = \sec^2(2x-2ct)\cdot(-2c),$$

$$\frac{\partial w}{\partial x} = \sec^2(2x-2ct)\cdot(2),$$

$$\frac{\partial^2 w}{\partial t^2} = (-2c)\cdot 2\sec(2x-2ct)\sec(2x-2ct)\tan(2x-2ct)\cdot(-2c)$$

$$= 8c^2\sec^2(2x-2ct)\tan(2x-2ct),$$

$$\frac{\partial^2 w}{\partial x^2} = (2)\cdot 2\sec(2x-2ct)\cdot\sec(2x-2ct)\tan(2x-2ct)\cdot(2)$$

$$= 8\sec^2(2x-2ct)\tan(2x-2ct),$$ then

$$\frac{\partial^2 w}{\partial t^2} = 8c^2\sec^2(2x-2ct)\tan(2x-2ct)$$

$$= c^2\left(8\sec^2(2x-2ct)\tan(2x-2ct)\right)$$

$$= c^2\cdot\frac{\partial^2 w}{\partial x^2}$$