Section 14.4

1.) $w = x^2 + 2Y, \quad x = \cos t, \quad Y = \sin t$

I.) $\frac{dw}{dt} = w_x \cdot \frac{dx}{dt} + w_Y \cdot \frac{dY}{dt}$

$= (2x) \cdot (-\sin t) + (2) \cdot (\cos t)$

$= (2\cos t)(-\sin t) + 2\cos t$

$= 2\cos t\,(1 - \sin t)$

OR

II.) $w = x^2 + 2Y = (\cos t)^2 + 2(\sin t) \xrightarrow{D}$

$\frac{dw}{dt} = 2(\cos t) \cdot (-\sin t) + 2 \cdot \cos t$

$= 2\cos t\,(1 - \sin t)$

if $t = \pi$, then $\frac{dw}{dt} = 2\cos\pi\,(1 - \sin\pi)$

$= 2(-1)(1-0) = -2$

3.) $w = \frac{x}{z} + \frac{Y}{z}, \quad x = \cos^2 t, \quad Y = \sin^2 t, \quad z = \frac{1}{t}$

I.) $\frac{dw}{dt} = w_x \cdot \frac{dx}{dt} + w_Y \cdot \frac{dY}{dt} + w_z \cdot \frac{dz}{dt}$

$= \frac{1}{z} \cdot \cancel{-2\cos t \sin t} + \frac{1}{z} \cdot \cancel{2\sin t \cos t}$

$+ \left(\frac{-x}{z^2} + \frac{-Y}{z^2}\right) \cdot \frac{-1}{t^2}$

$= t^2(-\cos^2 t - \sin^2 t) \cdot \frac{-1}{t^2}$

$= \underbrace{\cancel{(\cos^2 t + \sin^2 t)}}_{1} = 1$

II.) $w = \frac{x}{z} + \frac{Y}{z} = \frac{\cos^2 t}{1/t} + \frac{\sin^2 t}{1/t}$

$= t\cos^2 t + t\sin^2 t \xrightarrow{D}$

$$\frac{dw}{dt} = t \cdot -2\cancel{\cos t}\sin t + (1)\cos^2 t$$
$$+ t \cdot 2\cancel{\sin t}\cos t + (1)\sin^2 t = 1;$$
if $t = 3$, ~~then~~ $\frac{dw}{dt} = 1$.

5.) $w = 2Ye^X - \ln Z$, $X = \ln(t^2+1)$,
$Y = \arctan t$, $Z = e^t$

I.) $\frac{dw}{dt} = w_X \cdot \frac{dX}{dt} + w_Y \cdot \frac{dY}{dt} + w_Z \cdot \frac{dZ}{dt}$

$= 2Ye^X \cdot \frac{2t}{t^2+1} + 2e^X \cdot \frac{1}{t^2+1} + \frac{-1}{Z} \cdot e^t$

$= 2 \cdot \frac{\arctan t \cdot \cancel{(t^2+1)} \cdot 2t}{\cancel{t^2+1}}$

$+ 2\frac{\cancel{(t^2+1)}}{\cancel{t^2+1}} - \frac{1}{\cancel{e^t}}\cancel{e^t} = 4t\arctan t + 1$

II.) $w = 2Ye^X - \ln Z$

$= 2 \cdot \arctan t \cdot (t^2+1) - t \xrightarrow{D}$

$\frac{dw}{dt} = 2\arctan t \cdot 2t$
$+ 2 \cdot \frac{1}{\cancel{t^2+1}}\cancel{(t^2+1)} - 1 = 4t\arctan t + 1;$

if $t = 1$, then $\frac{dw}{dt} = 4(1)\arctan(1) + 1$

$= \cancel{4} \cdot \frac{\pi}{\cancel{4}} + 1 = \pi + 1$

6.) $w = z - \sin(xy), \quad x = t, \; y = \ln t, \; z = e^{t-1}$

I.) $\frac{dw}{dt} = w_x \cdot \frac{dx}{dt} + w_y \cdot \frac{dy}{dt} + w_z \cdot \frac{dz}{dt}$

$= -\cos(xy) \cdot y \cdot (1) + -\cos(xy) \cdot x \cdot \left(\frac{1}{t}\right) + (1) \cdot e^{t-1}$

$= -\cos(t \ln t) \cdot \ln t - \cos(t \ln t) \cdot \cancel{t}\left(\frac{1}{\cancel{t}}\right) + e^{t-1}$

$= -\cos(t \ln t) \cdot (\ln t + 1) + e^{t-1}$

OR

II.) $w = z - \sin(xy) = e^{t-1} - \sin(t \ln t) \xrightarrow{D}$

$\frac{dw}{dt} = e^{t-1} - \cos(t \ln t) \cdot \left[t \cdot \frac{1}{t} + (1) \ln t\right]$

$= e^{t-1} - \cos(t \ln t) \cdot [1 + \ln t]$;

if $t=1$, then $\frac{dw}{dt} = e^{0} - \cos(\overset{0}{\cancel{\ln 1}})\cdot[1+\overset{0}{\cancel{\ln 1}}]$

$= 1 - 1(1) = 0$

8.) $z = \arctan\left(\frac{x}{y}\right)$, $x = u\cos v$, $y = u\sin v$

I.) $\frac{\partial z}{\partial u} = z_x \cdot \frac{\partial x}{\partial u} + z_y \cdot \frac{\partial y}{\partial u}$

$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}\cdot\cos v + \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{-x}{y^2}\cdot\sin v$

$= \frac{y}{y^2+x^2}\cdot\cos v + \frac{-x}{y^2+x^2}\cdot\sin v$

$= \frac{u\sin v\cdot\cos v}{u^2\sin^2 v + u^2\cos^2 v} + \frac{-u\cos v\cdot\sin v}{u^2\sin^2 v + u^2\cos^2 v}$

$= 0$ OR

II.) $z = \arctan\left(\frac{x}{y}\right) = \arctan\left(\frac{\cancel{u}\cos v}{\cancel{u}\sin v}\right) \rightarrow$

$z = \arctan(\cot v) \xrightarrow{D}$

$\frac{\partial z}{\partial u} = 0$ (since v is constant);

if $(u,v) = (1.3, \pi/6)$, then $\frac{\partial z}{\partial u} = 0$;

I.) $\frac{\partial z}{\partial v} = z_x\cdot\frac{\partial x}{\partial v} + z_y\cdot\frac{\partial y}{\partial v}$

$= \frac{y}{y^2+x^2}\cdot -u\sin v + \frac{-x}{y^2+x^2}\cdot u\cos v$

$= \frac{u\sin v\cdot(-u\sin v)}{u^2\sin^2 v + u^2\cos^2 v} + \frac{-u\cos v\cdot(u\cos v)}{u^2\sin^2 v + u^2\cos^2 v}$

$= \frac{-u^2(\sin^2 v + \cos^2 v)}{u^2(\sin^2 v + \cos^2 v)} = -1$ OR

II.) $z = \arctan\left(\frac{x}{y}\right) = \arctan(\cot v) \xrightarrow{D}$

$$\frac{\partial z}{\partial v} = \frac{1}{1+(\cot v)^2} \cdot -\csc^2 v = \frac{-\csc^2 v}{\csc^2 v} = -1 \; ;$$

if $(u,v) = (1.3, 6)$, then $\frac{\partial z}{\partial v} = -1$.

9.) $w = xy + yz + xz$, $x = u+v$, $y = u-v$, $z = uv$

I.) $$\frac{\partial w}{\partial u} = w_x \cdot \frac{\partial x}{\partial u} + w_y \cdot \frac{\partial y}{\partial u} + w_z \cdot \frac{\partial z}{\partial u}$$

$$= (y+z)\cdot(1) + (x+z)\cdot(1) + (x+y)\cdot v$$

$$= (u-\cancel{v}) + uv + (u+\cancel{v}) + uv + ((u+v)+(u-v))\cdot v$$

$$= 2u + 2uv + 2uv = 2u + 4uv$$

OR

II.) $w = xy + yz + xz$

$$= (u+v)(u-v) + (u-v)(uv) + (u+v)(uv)$$

$$= u^2 - v^2 + u^2 v - \cancel{uv^2} + u^2 v + uv^2$$

$$= u^2 - v^2 + 2u^2 v \xrightarrow{D}$$

$$\frac{\partial w}{\partial u} = 2u + 4uv \; ; \text{ if } (u,v) = \left(\tfrac{1}{2}, 1\right),$$

~~then~~ $$\frac{\partial w}{\partial u} = 2\left(\tfrac{1}{2}\right) + 4\left(\tfrac{1}{2}\right)(1) = 1 + 2 = 3 \; ;$$

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I.) $$\frac{\partial w}{\partial v} = w_x \cdot \frac{\partial x}{\partial v} + w_y \cdot \frac{\partial y}{\partial v} + w_z \cdot \frac{\partial z}{\partial v}$$

$$= (y+z)(1) + (x+z)(-1) + (x+y)(u)$$

$$= (u-v) + uv + ((u+v) + uv)(-1) + ((u+\cancel{v}) + (u-\cancel{v}))(u)$$

$$= \cancel{u} - v + \cancel{uv} - \cancel{u} - v - \cancel{uv} + 2u^2$$

$$= 2u^2 - 2v$$

OR

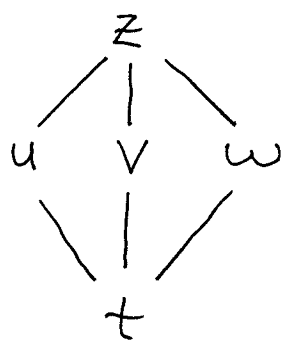
II.) $w = xy + yz + xz = u^2 - v^2 + 2u^2 v \xrightarrow{D}$

$$\frac{\partial w}{\partial v} = -2v + 2u^2 = 2u^2 - 2v \; ;$$

if $(u,v) = (\frac{1}{2}, 1)$, ~~then~~

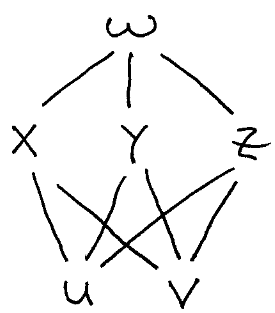
$$\frac{\partial w}{\partial v} = -2(1) + 2\left(\tfrac{1}{2}\right)^2 = -2 + \tfrac{1}{2} = {}^{-}3/2 \, .$$

14.)



$$\frac{dz}{dt} = f_u \cdot \frac{du}{dt} + f_v \cdot \frac{dv}{dt} + f_w \cdot \frac{dw}{dt}$$

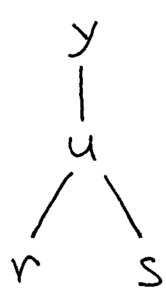
15.)



$$\frac{\partial w}{\partial u} = w_x \cdot \frac{\partial x}{\partial u} + w_y \cdot \frac{\partial y}{\partial u} + w_z \cdot \frac{\partial z}{\partial u}$$

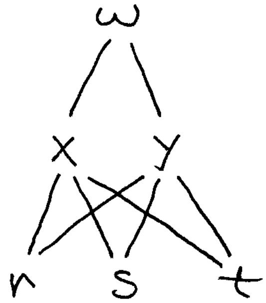
$$\frac{\partial w}{\partial v} = w_x \cdot \frac{\partial x}{\partial v} + w_y \cdot \frac{\partial y}{\partial v} + w_z \cdot \frac{\partial z}{\partial v}$$

20.)



$$\frac{\partial y}{\partial r} = \frac{dy}{du} \cdot \frac{\partial u}{\partial r}$$

24.)



$$\frac{\partial w}{\partial s} = w_x \cdot \frac{\partial x}{\partial s} + w_y \cdot \frac{\partial y}{\partial s}$$

26.) $\underbrace{xy + y^2 - 3x - 3}_{F(x,y)} = 0$;

By Theorem 8, $\frac{dy}{dx} = \frac{-F_x}{F_y} \rightarrow$

$\frac{dy}{dx} = \frac{-(y-3)}{x+2y}$ (Let $(x,y) = (-1,1)$.) $\rightarrow$

$\frac{dy}{dx} = \frac{-(1-3)}{-1+2(1)} = \frac{2}{1} = 2$

28.) $\underbrace{xe^y + \sin xy + y - \ln 2}_{F(x,y)} = 0$;

$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(e^y + \cos xy \cdot y)}{xe^y + \cos xy \cdot x + 1}$

(Let $(x,y) = (0, \ln 2)$.) $\rightarrow$

$\frac{dy}{dx} = \frac{-(e^{\ln 2} + \cancel{\cos} \overset{1}{0} \cdot \ln 2)}{0 + 0 + 1} = -2 - \ln 2$

30.) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ at $(2,3,6)$; $\rightarrow$ $\left(\frac{\partial}{\partial x}\right) \rightarrow$

$\frac{-1}{x^2} + (0) + \frac{-1}{z^2} \cdot z_x = 0 \rightarrow \frac{-1}{z^2} z_x = \frac{1}{x^2} \rightarrow$

$z_x = \frac{-z^2}{x^2}$, and $x = 2$, $z = 6 \rightarrow z_x = \frac{-36}{4}$

$\rightarrow$ $z_x = -9$

$$\to \left(\frac{\partial}{\partial Y}\right) \to (0) + \frac{-1}{Y^2} + \frac{-1}{Z^2}\cdot Z_Y = 0 \to$$

$$-\frac{1}{Z^2} Z_Y = \frac{1}{Y^2} \to Z_Y = \frac{-Z^2}{Y^2}, \text{ and}$$

$$Y = 3,\ Z = 6 \to Z_Y = \frac{-36}{9} \to Z_Y = -4$$

32.) $xe^Y + Ye^Z + 2\ln x - 2 - 3\ln 2 = 0$
at $(1, \ln 2, \ln 3)$; $\to \left(\frac{\partial}{\partial x}\right) \to$

$$e^Y + Y\cdot e^Z Z_x + 2\cdot\frac{1}{x} = 0, \text{ and } x = 1,\ Y = \ln 2,\ Z = \ln 3$$

$$\to 2 + (\ln 2)(3)\, Z_x + 2 = 0$$

$$\to 3(\ln 2)\, Z_x = -4 \to \boxed{Z_x = \frac{-4}{3\ln 2}};$$

$$\to \left(\frac{\partial}{\partial Y}\right) \to xe^Y + (Ye^Z Z_Y + (1)e^Z) = 0;$$

and $x = 1$, $Y = \ln 2$, $Z = \ln 3 \to$

$$(1)(2) + (\ln 2)(3)\, Z_Y + (3) = 0 \to$$

$$3(\ln 2)\, Z_Y = -5 \to \boxed{Z_Y = \frac{-5}{3\ln 2}}$$

39.)
$$\begin{tikzcd} & f \arrow[d] & \\ & x \arrow[dl] \arrow[dr] & \\ t & & s \end{tikzcd}$$

$w = f(x)$, $x = s^3 + t^2$, $f'(x) = e^x$;

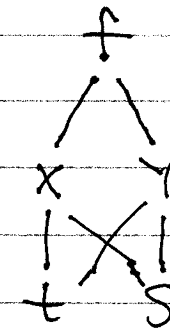
$$\frac{\partial w}{\partial t} = f'(x)\cdot\frac{\partial x}{\partial t}$$

$$= e^x \cdot 2t = 2t\cdot e^{s^3+t^2}$$

$$\frac{\partial w}{\partial s} = f'(x)\cdot\frac{\partial x}{\partial s} = e^{x}\cdot 3s^2 = 3s^2 e^{s^3+t^2}$$

40.) $w = f(x,y)$, $x = ts^2$, $y = \frac{s}{t}$,

$f_x = xy$, $f_y = \frac{1}{2}x^2$;



$$\frac{\partial w}{\partial t} = f_x\cdot\frac{\partial x}{\partial t} + f_y\cdot\frac{\partial y}{\partial t}$$

$$= xy\cdot(s^2) + \frac{1}{2}x^2\cdot\frac{-s}{t^2}$$

$$= (\cancel{t}s^2)\left(\frac{s}{\cancel{t}}\right)(s^2) + \frac{1}{2}(ts^2)^2\cdot\frac{-s}{t^2}$$

$$= s^5 + \frac{1}{2}\cancel{t^2}s^4\cdot\frac{-s}{\cancel{t^2}}$$

$$= s^5 - \frac{1}{2}s^5 = \frac{1}{2}s^5$$;

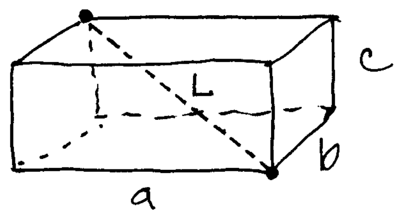
$$\frac{\partial w}{\partial s} = f_x\cdot\frac{\partial x}{\partial s} + f_y\cdot\frac{\partial y}{\partial s}$$

$$= xy\cdot(2ts) + \frac{1}{2}x^2\cdot\frac{1}{t}$$

$$= (ts^2)\left(\frac{s}{\cancel{t}}\right)(2\cancel{t}s) + \frac{1}{2}(ts^2)^2\cdot\frac{1}{t}$$

$$= 2ts^4 + \frac{1}{2}t^{\cancel{2}}s^4\cdot\frac{1}{\cancel{t}}$$

$$= 2ts^4 + \frac{1}{2}ts^4 = \boxed{\frac{5}{2}ts^4}$$

42.) 

Given

$\frac{da}{dt} = 1$ m./sec.,

$\frac{db}{dt} = 1$ m./sec., and $\frac{dc}{dt} = -3$ m./sec.

when $a = 1$ m., $b = 2$ m., and $c = 3$ m.

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volume $V = abc$; surface area

$S = 2ab + 2bc + 2ac$; diagonal

$L = \sqrt{a^2 + b^2 + c^2}$;

a.) Find $\frac{dV}{dt}$: (Use triple product rule.)

$$\frac{dV}{dt} = \frac{da}{dt} \cdot (bc) + \frac{db}{dt}(ac) + \frac{dc}{dt}(ab)$$

$$= (1)(2 \cdot 3) + (1)(1 \cdot 3) + (-3)(1 \cdot 2)$$

$$= 6 + 3 - 6 = +3 \text{ m.}^3/\text{sec.}$$

b.) Find $\frac{dS}{dt}$:

$$\frac{dS}{dt} = 2\left[\left(a \cdot \frac{db}{dt} + \frac{da}{dt} \cdot b\right) + \left(b \cdot \frac{dc}{dt} + \frac{db}{dt} \cdot c\right) + \left(a \cdot \frac{dc}{dt} + \frac{da}{dt} \cdot c\right)\right]$$

$$= 2\left[(1 \cdot 1 + 1 \cdot 2) + (2 \cdot (-3) + 1 \cdot 3) + (1 \cdot (-3) + 1 \cdot 3)\right]$$

$$= 2\left[3 + (-3) + (0)\right] = 2(0) = 0 \ m^2/sec.$$

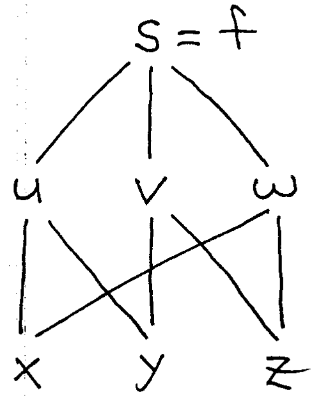
c.) Find $\frac{dL}{dt}$:

$$\frac{dL}{dt} = \frac{1}{\cancel{2}}(a^2 + b^2 + c^2)^{-1/2} \cdot \left[\cancel{2}a \frac{da}{dt} + \cancel{2}b \cdot \frac{db}{dt} + \cancel{2}c \cdot \frac{dc}{dt}\right]$$

$$= \frac{1}{\sqrt{14}}\left[1 \cdot 1 + 2 \cdot 1 + 3 \cdot (-3)\right] = \frac{-6}{\sqrt{14}} \ m./sec.$$

(The diagonal is $\downarrow$.)

43.) assume function $s = f(u,v,w)$ and $u = x - y$, $v = y - z$, $w = z - x$.



Show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0 :$$

Using the chain rule $\longrightarrow$

$$\frac{\partial f}{\partial x} = s_u \cdot \frac{\partial u}{\partial x} + s_w \cdot \frac{\partial w}{\partial x}$$
$$= s_u \cdot (1) + s_w \cdot (-1) = s_u - s_w \ ;$$

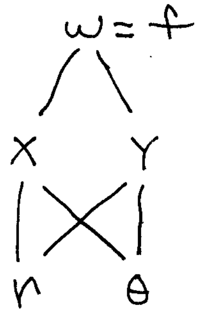
$$\frac{\partial f}{\partial y} = s_u \cdot \frac{\partial u}{\partial y} + s_v \cdot \frac{\partial v}{\partial y}$$
$$= s_u \cdot (-1) + s_v \cdot (1) = s_v - s_u \ ;$$

$$\frac{\partial f}{\partial z} = s_v \cdot \frac{\partial v}{\partial z} + s_w \cdot \frac{\partial w}{\partial z}$$
$$= s_v \cdot (-1) + s_w \cdot (1) = s_w - s_v \ ;$$

then

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = (\cancel{s_u} - \cancel{s_w}) + (\cancel{s_v} - \cancel{s_u}) + (\cancel{s_w} - \cancel{s_v}) = 0 .$$

44.) assume function $w = f(x,y)$ and $x = r\cos\theta$, $y = r\sin\theta$. Then



a.) $\dfrac{\partial w}{\partial r} = f_X \cdot \dfrac{\partial X}{\partial r} + f_Y \cdot \dfrac{\partial Y}{\partial r}$

$= f_X \cdot (\cos\theta) + f_Y \cdot (\sin\theta)$;

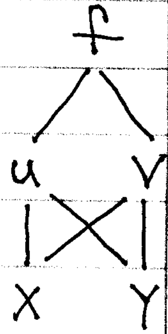
$\dfrac{\partial w}{\partial \theta} = f_X \cdot \dfrac{\partial X}{\partial \theta} + f_Y \cdot \dfrac{\partial Y}{\partial \theta}$

$= f_X \cdot (-r\sin\theta) + f_Y \cdot (r\cos\theta)$

$= r\,(-f_X \cdot \sin\theta + f_Y \cdot \cos\theta) \longrightarrow$

$\dfrac{1}{r}\dfrac{\partial w}{\partial \theta} = -f_X \cdot \sin\theta + f_Y \cdot \cos\theta$.

45.) $w = f(u,v)$, $u = \frac{1}{2}(X^2 - Y^2)$, $V = XY$;



$\dfrac{\partial w}{\partial X} = f_u \cdot \dfrac{\partial u}{\partial X} + f_v \cdot \dfrac{\partial v}{\partial X}$

$= f_u \cdot \frac{1}{2}(2X) + f_V \cdot (Y)$

$= X f_u + Y f_V$,

$\dfrac{\partial^2 w}{\partial X^2} = X \cdot \left[f_{uu} \dfrac{\partial u}{\partial X} + f_{uv} \cdot \dfrac{\partial v}{\partial X} \right] + (1) f_u$

$+ Y \left[f_{vu} \cdot \dfrac{\partial u}{\partial X} + f_{vv} \dfrac{\partial v}{\partial X} \right]$

$= X\left[f_{uu} \cdot X + f_{uv} \cdot Y \right] + f_u$

$+ Y\left[f_{uv} \cdot X + f_{vv} \cdot Y \right]$

$$= x^2 f_{uu} + xy f_{uv} + xy f_{uv} + y^2 f_{vv} + f_u$$

$$= \underline{x^2 f_{uu} + 2xy f_{uv} + y^2 f_{vv} + f_u} \; ;$$

$$\frac{\partial w}{\partial y} = f_u \cdot \frac{\partial u}{\partial y} + f_v \cdot \frac{\partial v}{\partial y}$$

$$= f_u \cdot (-y) + f_v \cdot (x) = -y f_u + x f_v \, ,$$

$$\frac{\partial^2 w}{\partial y^2} = -y \cdot \left[f_{uu} \cdot \frac{\partial u}{\partial y} + f_{uv} \cdot \frac{\partial v}{\partial y} \right] + (-1) f_u$$

$$+ x \cdot \left[f_{vu} \cdot \frac{\partial u}{\partial y} + f_{vv} \cdot \frac{\partial v}{\partial y} \right]$$

$$= -y \left[f_{uu} \cdot (-y) + f_{uv} \cdot (x) \right] - f_u$$

$$+ x \left[f_{uv} \cdot (-y) + f_{vv} \cdot (x) \right]$$

$$= y^2 \cdot f_{uu} - xy f_{uv} - xy f_{uv} + x^2 f_{vv} - f_u$$

$$= \underline{y^2 f_{uu} - 2xy f_{uv} + x^2 f_{vv} - f_u} \; ;$$

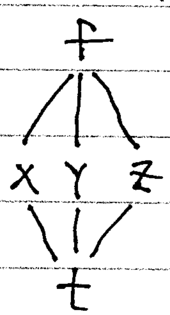
then $w_{xx} + w_{yy} = x^2 f_{uu} + \cancel{2xy} f_{uv} + y^2 f_{vv} + \cancel{f_u}$

$$+ y^2 f_{uu} - \cancel{2xy} f_{uv} + x^2 f_{vv} - \cancel{f_u}$$

$$= x^2 (f_{uu} + f_{vv}) + y^2 (f_{uu} + f_{vv})$$

$$= x^2 (0) + y^2 (0) = 0$$

47.) $w = f(x, y, z)$, $x = \cos t$, $y = \sin t$, $z = t$ and $f_x = \cos t$, $f_y = \sin t$, $f_z = t^2 + t - 2$; f has extrema when $\frac{dw}{dt} = 0$:

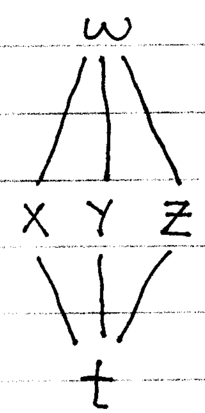


$$\frac{dw}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt} + f_z \cdot \frac{dz}{dt}$$

$$= \cancel{(\cos t)(-\sin t)} + \cancel{(\sin t)(\cos t)} + (t^2+t-2)(1)$$

$$= (t-1)(t+2) = 0 \longrightarrow t=1,\ t=-2$$

48.) $w = x^2 e^{2y} \cos 3z$, $x = \cos t$, $y = \ln(t+2)$, $z = t$; then



$$\frac{dw}{dt} = w_x \cdot \frac{dx}{dt} + w_y \cdot \frac{dy}{dt} + w_z \cdot \frac{dz}{dt}$$

$$= (2x e^{2y} \cos 3z)(-\sin t) + (x^2 \cdot 2e^{2y} \cos 3z)\left(\frac{1}{t+2}\right) + (x^2 e^{2y} \cdot -3 \sin 3z)\cdot(1)$$

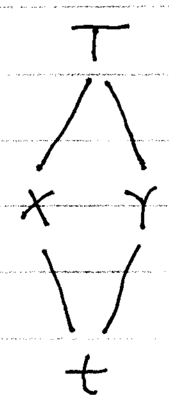
and $x=1$, $y=\ln 2$, $z=0$, $t=0$

$$= (2e^{2\ln 2} \cdot \cos 0)(-\sin 0) + (2e^{2\ln 2} \cdot \cos 0)\left(\frac{1}{0+2}\right) + (e^{2\ln 2} \cdot -3 \sin 0)$$

$$= e^{\ln 2^2} = 4$$

49.) $T = f(x,y)$, $x = \cos t$, $y = \sin t$ and $T_x = 8x - 4y$, $T_y = 8y - 4x$;

a.) max/min occur when $\frac{dT}{dt} = 0$:



$$\frac{dT}{dt} = T_x \cdot \frac{dx}{dt} + T_y \cdot \frac{dy}{dt}$$

$$= (8x - 4y)(-\sin t) + (8y - 4x)(\cos t)$$

$$= (8\cos t - 4\sin t)(-\sin t) + (8\sin t - 4\cos t)(\cos t)$$

$= -8\cos t \cancel{\sin t} + 4\sin^2 t$
$\quad + 8 \cancel{\cos t} \sin t - 4\cos^2 t$

$= -4(\cos^2 t - \sin^2 t)$

$= -4\cos 2t = 0 \quad$ on $[0, 2\pi]$

$\rightarrow 2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$;

$\frac{dT}{dt} = -4\cos 2t \rightarrow$

$\frac{d^2T}{dt^2} = -4 \cdot {}^-2\sin 2t = 8\sin 2t$

$\underline{\underline{t = \frac{\pi}{4}}}$: $\frac{d^2T}{dt^2} = 8\sin\frac{\pi}{2} = 8(1) = 8 > 0$

so min T at $t = \frac{\pi}{4}$, $X = \frac{\sqrt{2}}{2}$, $Y = \frac{\sqrt{2}}{2}$;

$\underline{\underline{t = \frac{3\pi}{4}}}$: $\frac{d^2T}{dt^2} = 8\sin\frac{3\pi}{2} = 8(-1) = -8 < 0$

so max. T at $t = \frac{3\pi}{4}$, $X = {}^-\frac{\sqrt{2}}{2}$, $Y = \frac{\sqrt{2}}{2}$;

$t = \frac{5\pi}{4}$: $\frac{d^2T}{dt^2} = 8\sin\frac{5\pi}{2} = 8(1) = 8 > 0$

so min. T at $t = \frac{5\pi}{4}$, $X = {}^-\frac{\sqrt{2}}{2}$, $Y = {}^-\frac{\sqrt{2}}{2}$;

$t = \frac{7\pi}{4}$: $\frac{d^2T}{dt^2} = 8\sin\frac{7\pi}{2} = 8(-1) = -8 < 0$

so max T at $t = \frac{7\pi}{4}$, $X = \frac{\sqrt{2}}{2}$, $Y = {}^-\frac{\sqrt{2}}{2}$.

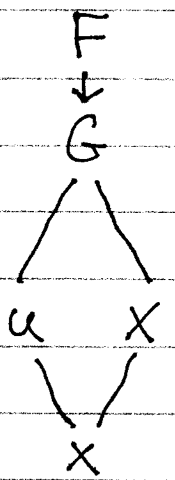
FACT: $F(x) = \int_a^b g(t,x)\,dt \xrightarrow{D}$

$$F'(x) = \int_a^b g_x(t,x)\,dt.$$

FACT: $G(u,x) = \int_a^u g(t,x)\,dt \xrightarrow{D}$

$$G_u(u,x) = g(u,x) \quad \text{and}$$

$$G_x(u,x) = \int_a^u g_x(t,x)\,dt$$



51.) $F(x) = \int_0^{x^2} \sqrt{t^4 + x^3}\,dt$ can be written as

$$G(u,x) = \int_0^u \sqrt{t^4 + x^3}\,dt$$

where $u = x^2$; then

$$F'(x) = \frac{d}{dx} G(u,x)$$

$$= G_u(u,x)\cdot\frac{du}{dx} + G_x(u,x)\cdot\frac{dx}{dx}$$

$$= g(u,x)\cdot(2x) + \int_0^u g_x(t,x)\,dt \cdot (1)$$

$$= \sqrt{u^4 + x^3}\cdot(2x) + \int_0^u \frac{1}{2}(t^4 + x^3)^{-1/2}\cdot 3x^2\,dt$$

$$= 2x\sqrt{(x^2)^4 + x^3} + \frac{3}{2}\int_0^{x^2} x^2 (t^4 + x^3)^{-1/2}\,dt$$

$$= 2x\sqrt{x^8 + x^3} + \frac{3}{2}\int_0^{x^2} \frac{x^2}{\sqrt{t^4 + x^3}}\,dt$$

52.) $F(x) = \int_{x^2}^{1} \sqrt{t^3 + x^2}\, dt$

$= -\int_{1}^{x^2} \sqrt{t^3 + x^2}\, dt$ can be

written as

$G(u, x) = \int_{1}^{u} -\sqrt{t^3 + x^2}\, dt$

where $u = x^2$; then

$F'(x) = \frac{d}{dx} G(u, x)$

$= G_u(u, x) \cdot \frac{du}{dx} + G_x(u, x) \cdot \frac{dx}{dx}$

$= g(u, x) \cdot (2x) + \int_{1}^{u} g_x(t, x)\, dt$

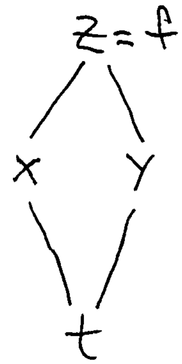
$= -\sqrt{u^3 + x^2} \cdot (2x) + \int_{1}^{u} -\frac{1}{2}(t^3 + x^2)^{-1/2} \cdot (2x)\, dt$

$= -2x\sqrt{(x^2)^3 + x^2} - \int_{1}^{x^2} x\,(t^3 + x^2)^{-1/2}\, dt$

$= -2x\sqrt{x^6 + x^2} - \int_{1}^{x^2} \frac{x}{\sqrt{t^3 + x^2}}\, dt$

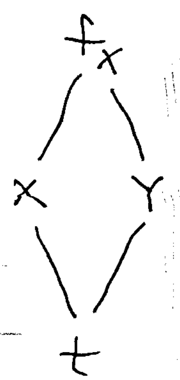
1.) a.) assume $z = f(x, y)$ and $x = e^{2t}$, $y = \sin t$.

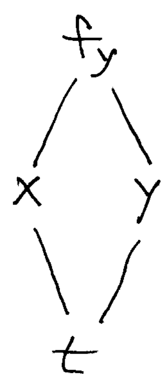
Then by the chain rule



$$\frac{dz}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$$

$$= f_x \cdot 2e^{2t} + f_y \cdot \cos t \quad ; \text{ and}$$





$$\frac{d^2z}{dt^2} = \frac{d}{dt}\left(\frac{dz}{dt}\right)$$

$$= \frac{d}{dt}\left[f_x \cdot 2e^{2t} + f_y \cdot \cos t\right]$$

$$= f_x \cdot \frac{d}{dt}\left(2e^{2t}\right) + \frac{d}{dt}\left(f_x\right) \cdot 2e^{2t}$$

$$+ f_y \cdot \frac{d}{dt}(\cos t) + \frac{d}{dt}\left(f_y\right) \cdot \cos t$$

$$= f_x \cdot 4e^{2t} + \left[f_{xx} \cdot \frac{dx}{dt} + f_{xy} \cdot \frac{dy}{dt} \right] \cdot 2e^{2t}$$

$$+ f_y \cdot (-\sin t) + \left[f_{yx} \cdot \frac{dx}{dt} + f_{yy} \cdot \frac{dy}{dt} \right] \cdot \cos t$$

$$= f_x \cdot 4e^{2t} + \left[f_{xx} \cdot 2e^{2t} + f_{xy} \cdot \cos t \right] \cdot 2e^{2t}$$

$$- f_y \cdot \sin t + \left[f_{xy} \cdot 2e^{2t} + f_{yy} \cdot \cos t \right] \cdot \cos t$$

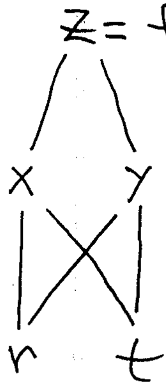
$$= f_x \cdot (4e^{2t}) - f_y \cdot (\sin t)$$

$$+ f_{xx} \cdot (4e^{4t}) + f_{yy} \cdot (\cos^2 t)$$

$$+ f_{xy} \cdot (4e^{2t} \cos t) \quad .$$

1.) 6.) assume $z = f(x, y)$ and $x = rt^2$, $y = r^3 - t$.

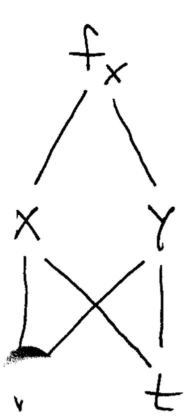
Then by chain rule

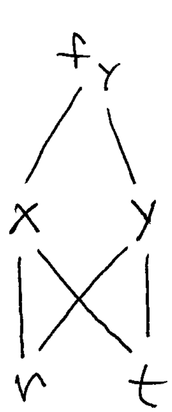


i.) $$\frac{\partial z}{\partial t} = f_x \cdot \frac{\partial x}{\partial t} + f_y \cdot \frac{\partial y}{\partial t}$$

$$= f_x \cdot (2rt) + f_y \cdot (-1)$$

$$= f_x \cdot (2rt) - f_y \quad ; \text{ then}$$





$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t} \left[\frac{\partial z}{\partial t} \right]$$

$$= \frac{\partial}{\partial t} \left[f_x \cdot (2rt) - f_y \right]$$

$$= f_x \cdot \frac{\partial}{\partial t}(2rt) + \frac{\partial}{\partial t}(f_x) \cdot (2rt)$$

$$- \frac{\partial}{\partial t}(f_y)$$

$$= f_x \cdot 2r + \left[f_{xx} \cdot \frac{\partial x}{\partial t} + f_{xy} \cdot \frac{\partial y}{\partial t} \right] \cdot (2rt)$$

$$- \left[f_{yx} \cdot \frac{\partial x}{\partial t} + f_{yy} \cdot \frac{\partial y}{\partial t} \right]$$

$$= f_x \cdot 2r + f_{xx} \cdot (2rt)(2rt) + f_{xy} \cdot (-1)(2rt)$$

$$- f_{xy} \cdot (2rt) - f_{yy} \cdot (-1)$$

$$= f_x \cdot (2r) + f_{xx} \cdot (4r^2t^2)$$

$$- f_{xy} \cdot (4rt) + f_{yy}$$

ii.) $\dfrac{\partial z}{\partial r} = f_x \cdot \dfrac{\partial x}{\partial r} + f_y \cdot \dfrac{\partial y}{\partial r}$

$$= f_x \cdot t^2 + f_y \cdot 3r^2 \quad ; \text{ then}$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left[\frac{\partial z}{\partial r} \right] = \frac{\partial}{\partial r}\left[f_x \cdot t^2 + f_y \cdot 3r^2 \right]$$

$$= \frac{\partial}{\partial r}\left[f_x \right] \cdot t^2 + f_y \cdot \frac{\partial}{\partial r}(3r^2) + \frac{\partial}{\partial r}(f_y) \cdot 3r^2$$

$$= \left[f_{xx} \cdot \frac{\partial x}{\partial r} + f_{xy} \cdot \frac{\partial y}{\partial r} \right] \cdot t^2$$

$$+ f_y \cdot 6r + \left[f_{yx} \cdot \frac{\partial x}{\partial r} + f_{yy} \cdot \frac{\partial y}{\partial r} \right] \cdot 3r^2$$

$$= f_{xx} \cdot (t^2)(t^2) + f_{xy} (3r^2)(t^2)$$

$$+ f_y \cdot 6r + f_{xy} \cdot (t^2)(3r^2) + f_{yy} \cdot (3r^2)(3r^2)$$

$$= f_{xx} \cdot (t^4) + f_y \cdot (6r) + f_{yy} \cdot (9r^4)$$

$$+ f_{xy} \cdot (6r^2t^2)$$