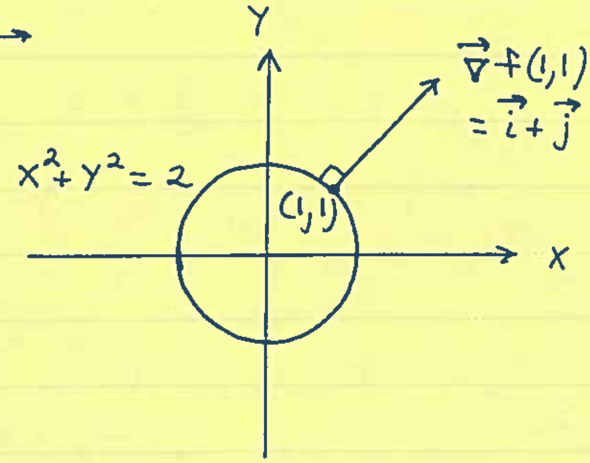
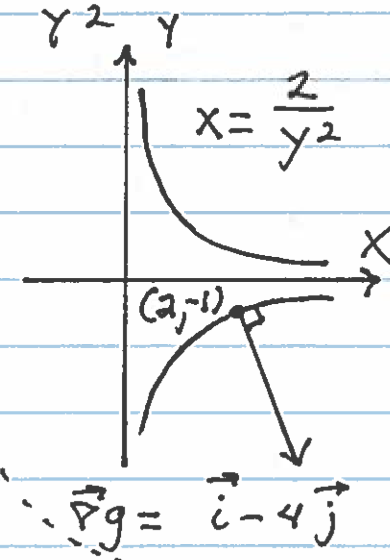
Section 14.5

2.) $f(x,y) = \ln(x^2+y^2)$ and $P = (1,1)$;
$f(1,1) = \ln 2$ then level curve at $(1,1)$
is $\ln 2 = \ln(x^2+y^2) \rightarrow$
$\underline{x^2+y^2=2}$; and
$f_x = \dfrac{2x}{x^2+y^2}$ and
$f_y = \dfrac{2y}{x^2+y^2}$ so
gradient vector is
$\vec{\nabla} f(1,1) = f_x(1,1)\vec{i} + f_y(1,1)\vec{j}$
$= (1)\vec{i} + (1)\vec{j} = \vec{i}+\vec{j}$

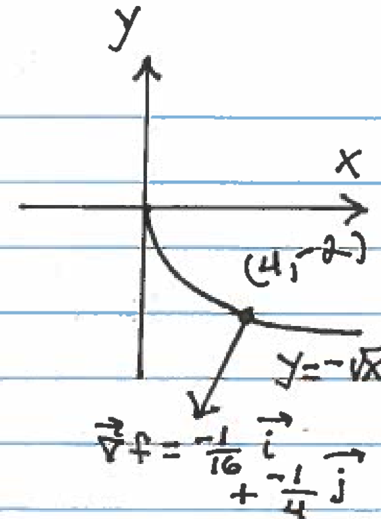


3.) $g(x,y) = xy^2$ and $P = (2,-1)$;
$g(2,-1) = 2(-1)^2 = 2$ then level curve
at $(2,-1)$ is $2 = xy^2 \rightarrow x = \dfrac{2}{y^2}$:
and $g_x = y^2$, $g_y = 2xy$ so
gradient vector is
$\vec{\nabla} g(2,-1) = g_x(2,-1)\vec{i} + g_y(2,-1)\vec{j}$
$= (1)\vec{i} + (-4)\vec{j}$



6.) $f(x,y) = \arctan\left(\dfrac{\sqrt{x}}{y}\right)$ and
$P = (4,-2)$;
$f(4,-2) = \arctan\left(\dfrac{\sqrt{4}}{-2}\right) = \arctan(-1)$
$= -\pi/4$
then level curve at $(4,-2)$ is
$-\dfrac{\pi}{4} = \arctan\left(\dfrac{\sqrt{x}}{y}\right) \rightarrow \dfrac{\sqrt{x}}{y} = -1 \rightarrow$

$y = -\sqrt{x}$:



and $f_x = \dfrac{1}{1+\left(\frac{\sqrt{x}}{y}\right)^2} \cdot \dfrac{1}{y} \cdot \dfrac{1}{2} x^{-1/2}$

$= \dfrac{1}{1+\frac{x}{y^2}} \cdot \dfrac{1}{2\sqrt{x}\, y} \cdot \dfrac{y}{y} = \dfrac{y}{2\sqrt{x}\, y^2 + 2x^{3/2}}$,

and $f_y = \dfrac{1}{1+\left(\frac{\sqrt{x}}{y}\right)^2} \cdot \dfrac{-\sqrt{x}}{y^2} = \dfrac{-\sqrt{x}}{y^2+x}$, so

gradient vector is

$\vec{\nabla} f(4,-2) = f_x(4,-2)\,\vec{i} + f_y(4,-2)\,\vec{j}$

$= \dfrac{-2}{16+16}\,\vec{i} + \dfrac{-2}{4+4}\,\vec{j} = \dfrac{-1}{16}\,\vec{i} + \dfrac{-1}{4}\,\vec{j}$

7.) $f(x,y,z) = x^2 + y^2 - 2z^2 + z \ln x$
and $P = (1,1,1)$; then
$f_x = 2x + z \cdot \dfrac{1}{x}$, $f_y = 2y$, $f_z = -4z + \ln x$
so that the gradient vector is
$\vec{\nabla} f(1,1,1) = f_x(1,1,1)\,\vec{i} + f_y(1,1,1)\,\vec{j} + f_z(1,1,1)\,\vec{k}$

$= 3\,\vec{i} + 2\,\vec{j} - 4\,\vec{k}$

10.) $f(x,y,z) = e^{x+y}\cos z + (y+1)\arcsin x$ and $P = (0,0,\pi/6)$; then

$$f_x = e^{x+y}\cos z + (y+1)\cdot\frac{1}{\sqrt{1-x^2}} ,$$

$$f_y = e^{x+y}\cos z + \arcsin x ,$$

$$f_z = -e^{x+y}\cdot \sin z ,$$ so that the gradient vector is

$$\vec{\nabla} f(0,0,\tfrac{\pi}{6}) = f_x(0,0,\tfrac{\pi}{6})\,\vec{i} + f_y(0,0,\tfrac{\pi}{6})\,\vec{j} + f_z(0,0,\tfrac{\pi}{6})\,\vec{k}$$

$$= \left(\frac{\sqrt{3}}{2}+1\right)\vec{i} + \frac{\sqrt{3}}{2}\,\vec{j} - \frac{1}{2}\,\vec{k}$$

12.) $f(x,y) = 2x^2 + y^2$, $P = (-1,1)$, and $\vec{A} = 3\vec{i} - 4\vec{j}$; then direction vector

$$\vec{u} = \frac{1}{|\vec{A}|}\vec{A} = \frac{1}{5}(3\vec{i} - 4\vec{j}) = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$$; and

$f_x = 4x$, $f_y = 2y$, so gradient vector

$$\vec{\nabla} f(-1,1) = f_x(-1,1)\,\vec{i} + f_y(-1,1)\,\vec{j} = -4\vec{i} + 2\vec{j}$$, then

$$D_{\vec{u}} f(-1,1) = \vec{\nabla} f(-1,1)\cdot\vec{u}$$

$$= (-4\vec{i} + 2\vec{j})\cdot\left(\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}\right)$$

$$= \frac{-12}{5} - \frac{8}{5} = -4$$

13.) $g(x,y) = \dfrac{x-y}{xy+2}$, $P = (1,-1)$,

$\vec{A} = 12\vec{i} + 5\vec{j}$, then direction vector

$\vec{u} = \dfrac{1}{|\vec{A}|}\vec{A} = \dfrac{1}{13}(12\vec{i} + 5\vec{j}) = \dfrac{12}{13}\vec{i} + \dfrac{5}{13}\vec{j}$;

and $g_x = \dfrac{(xy+2)(1) - (x-y)y}{(xy+2)^2}$

$= \dfrac{\cancel{xy} + 2 - \cancel{xy} + y^2}{(xy+2)^2} = \dfrac{2+y^2}{(xy+2)^2}$,

$g_y = \dfrac{(xy+2)(-1) - (x-y)(x)}{(xy+2)^2}$

$= \dfrac{-\cancel{xy} - 2 - x^2 + \cancel{xy}}{(xy+2)^2} = \dfrac{-2-x^2}{(xy+2)^2}$;

so gradient vector is

$\vec{\nabla} g(1,-1) = g_x(1,-1)\vec{i} + g_y(1,-1)\vec{j}$

$= \dfrac{3}{1}\vec{i} + \dfrac{-3}{1}\vec{j} = 3\vec{i} - 3\vec{j}$, then

$D_{\vec{u}}\, g(1,-1) = \vec{\nabla} g(1,-1) \cdot \vec{u}$

$= (3\vec{i} - 3\vec{j}) \cdot \left(\dfrac{12}{13}\vec{i} + \dfrac{5}{13}\vec{j}\right)$

$= \dfrac{36}{13} - \dfrac{15}{13} = \dfrac{21}{13}$

$f(x,y,z) = xy + yz + xz$, $P = (1,-1,2)$, and

15.) $\vec{A} = 3\vec{i} + 6\vec{j} - 2\vec{k}$; then direction vector

$$\vec{u} = \frac{1}{|\vec{A}|}\vec{A} = \frac{1}{7}\left(3\vec{i} + 6\vec{j} - 2\vec{k}\right) = \frac{3}{7}\vec{i} + \frac{6}{7}\vec{j} - \frac{2}{7}\vec{k} \text{ ;}$$

and $f_x = y + z$, $f_y = x + z$, $f_z = x + y$, so gradient vector is

$$\vec{\nabla} f(1,-1,2) = f_x(1,-1,2)\vec{i} + f_y(1,-1,2)\vec{j} + f_z(1,-1,2)\vec{k}$$
$$= (1)\vec{i} + (3)\vec{j} + (0)\vec{k} = \vec{i} + 3\vec{j} \text{ ; then}$$

$$D_{\vec{u}} f(1,-1,2) = \vec{\nabla} f(1,-1,2) \cdot \vec{u}$$
$$= (\vec{i} + 3\vec{j}) \cdot \left(\frac{3}{7}\vec{i} + \frac{6}{7}\vec{j} - \frac{2}{7}\vec{k}\right)$$
$$= \frac{3}{7} + \frac{18}{7} = 3 \text{ .}$$

16.) $f(x,y,z) = x^2 + 2y^2 - 3z^2$, $P = (1,1,1)$, and $\vec{A} = \vec{i} + \vec{j} + \vec{k}$; then direction vector

$$\vec{u} = \frac{1}{|\vec{A}|}\vec{A} = \frac{1}{\sqrt{3}}\left(\vec{i} + \vec{j} + \vec{k}\right) = \frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k} \text{ ;}$$

and $f_x = 2x$, $f_y = 4y$, $f_z = -6z$, so gradient vector is

$$\vec{\nabla} f(1,1,1) = f_x(1,1,1)\vec{i} + f_y(1,1,1)\vec{j} + f_z(1,1,1)\vec{k}$$
$$= (2)\vec{i} + (4)\vec{j} + (-6)\vec{k} = 2\vec{i} + 4\vec{j} - 6\vec{k} \text{ ;}$$

then $D_{\vec{u}} f(1,1,1) = \vec{\nabla} f(1,1,1) \cdot \vec{u}$

$$= (2\vec{i} + 4\vec{j} - 6\vec{k}) \cdot \left(\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}\right)$$
$$= \frac{2}{\sqrt{3}} + \frac{4}{\sqrt{3}} + \frac{-6}{\sqrt{3}} = 0 \text{ .}$$

17.) $g(x,y,z) = 3e^x \cos yz$, $P = (0,0,0)$, and $\vec{A} = 2\vec{i} + \vec{j} - 2\vec{k}$; then direction vector

$$\vec{u} = \frac{1}{|\vec{A}|}\vec{A} = \frac{1}{3}(2\vec{i} + \vec{j} - 2\vec{k})$$
$$= \frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k} \text{ ; and}$$

$g_x = 3e^x \cos yz$, $g_y = -3e^x z \sin yz$, $g_z = -3e^x y \sin yz$, so gradient vector is

$$\vec{\nabla} g(0,0,0) = g_x(0,0,0)\vec{i} + g_y(0,0,0)\vec{j} + g_z(0,0,0)\vec{k}$$
$$= (3)\vec{i} + (0)\vec{j} + (0)\vec{k} = 3\vec{i} \text{ ;}$$

then $D_{\vec{u}}\, g(0,0,0) = \vec{\nabla} g(0,0,0) \cdot \vec{u}$

$$= (3\vec{i}) \cdot \left(\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}\right)$$
$$= 3\left(\frac{2}{3}\right) = 2$$

19.) $f(x,y) = x^2 + xy + y^2$ and $P = (-1,1)$;
$f_x = 2x + y$, $f_y = x + 2y$ so gradient vector is

$$\vec{\nabla} f(-1,1) = f_x(-1,1)\,\vec{i} + f_y(-1,1)\,\vec{j}$$
$$= (-1)\,\vec{i} + (1)\,\vec{j} = -\vec{i} + \vec{j} \;;$$

a.) The direction of maximum increase is in the same direction as $\vec{\nabla} f(-1,1)$, i.e.,

$$\vec{u} = \frac{1}{|\vec{\nabla} f(-1,1)|}\,\vec{\nabla} f(-1,1) = \frac{1}{\sqrt{2}}(-\vec{i} + \vec{j}) = \frac{-1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j} \;;$$

the dir. der. in this direction is

$$D_{\vec{u}} f(-1,1) = |\vec{\nabla} f(-1,1)| = \sqrt{2} \;.$$

b.) The direction of maximum decrease is in the opposite direction of $\vec{\nabla} f(-1,1)$, i.e., $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$; the dir. der. in this direction is

$$D_{\vec{u}} f(-1,1) = -|\vec{\nabla} f(-1,1)| = -\sqrt{2}$$

22.) $g(x,y,z) = xe^y + z^2$ and $P = (1, \ln 2, 1/2)$;
$g_x = e^y$, $g_y = xe^y$, $g_z = 2z$ so gradient vector is

$$\vec{\nabla} g(1, \ln 2, 1/2) = g_x(1,\ln 2,\tfrac{1}{2})\,\vec{i} + g_y(1,\ln 2,\tfrac{1}{2})\,\vec{j} + g_z(1,\ln 2,\tfrac{1}{2})\,\vec{k}$$
$$= (2)\,\vec{i} + (2)\,\vec{j} + (1)\,\vec{k} = 2\vec{i} + 2\vec{j} + \vec{k}$$

a.) The direction of maximum increase is in the same direction as $\vec{\nabla} g(1, \ln 2, \frac{1}{2})$, i.e., $\vec{u} = \frac{1}{|\vec{\nabla} g(1, \ln 2, \frac{1}{2})|} \vec{\nabla} g(1, \ln 2, \frac{1}{2})$

$= \frac{1}{3}(2\vec{i} + 2\vec{j} + \vec{k}) = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$;

the dir. der. in this direction is

$D_{\vec{u}}\, g(1, \ln 2, \frac{1}{2}) = |\vec{\nabla} g(1, \ln 2, \frac{1}{2})| = 3$.

b.) The direction of maximum <u>decrease</u> is in the <u>opposite</u> direction of $\vec{\nabla} g(1, \ln 2, \frac{1}{2})$, i.e., $\vec{u} = -\frac{2}{3}\vec{i} - \frac{2}{3}\vec{j} - \frac{1}{3}\vec{k}$; the dir. der. in this direction is

$D_{\vec{u}}\, g(1, \ln 2, \frac{1}{2}) = -|\nabla g(1, \ln 2, \frac{1}{2})| = -3$.

29.) $f(x,y) = x^2 - xy + y^2 - y$ and $P = (1,-1)$;

$f_x = 2x - y$, $f_y = -x + 2y - 1$ so gradient vector is

$\vec{\nabla} f(1,-1) = f_x(1,-1)\,\vec{i} + f_y(1,-1)\,\vec{j}$

$= 3\,\vec{i} + (-4)\,\vec{j}$; let unit vector $\vec{u} = a\vec{i} + b\vec{j}$, where $a^2 + b^2 = 1$; then:

a.) $D_{\vec{u}} f(1,-1)$ is largest when $\vec{u}$ is in the direction of the gradient vector ; so $\vec{u} = \frac{1}{|\vec{\nabla} f(1,-1)|} \vec{\nabla} f(1,-1)$

$= \frac{1}{5}(3\vec{i} - 4\vec{j}) = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$ and

$D_{\vec{u}} f(1,-1) = \vec{\nabla} f(1,-1) \cdot \vec{u}$

$= (3\vec{i} - 4\vec{j}) \cdot \left(\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}\right)$

$= \frac{9}{5} + \frac{16}{5} = \frac{25}{5} = 5$

b.) $D_{\vec{u}} f(1,-1)$ is smallest when $\vec{u}$ is in the direction of the negative gradient vector ; so

$\vec{u} = \frac{-1}{|-\vec{\nabla} f(1,-1)|} \vec{\nabla} f(1,-1) = -\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$

and $D_{\vec{u}} f(1,-1) = \vec{\nabla} f(1,-1) \cdot \vec{u}$

$= (3\vec{i} - 4\vec{j}) \cdot \left(-\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}\right)$

$= \frac{-9}{5} + \frac{-16}{5} = \frac{-25}{5} = -5$

c.) $D_{\vec{u}} f(1,-1) = 0 \rightarrow$

$\vec{\nabla} f(1,-1) \cdot \vec{u} = (3\vec{i} - 4\vec{j}) \cdot (a\vec{i} + b\vec{j})$

$\left.\begin{aligned} &= 3a - 4b = 0 \\ \text{and} \quad &a^2 + b^2 = 1 \end{aligned}\right\} \rightarrow 3a = 4b \rightarrow$

$\underline{a = \frac{4}{3}b} \rightarrow$ (SUB)

$\left(\frac{4}{3}b\right)^2 + b^2 = 1 \rightarrow \frac{16}{9}b^2 + b^2 = 1 \rightarrow$

$16b^2 + 9b^2 = 9 \rightarrow 25b^2 = 9 \rightarrow$

$b^2 = \frac{9}{25} \rightarrow b = \pm\frac{3}{5}$;

if $b = \frac{3}{5}$, then $a = \frac{4}{3}\left(\frac{3}{5}\right) = 4/5$

so $\boxed{\vec{u} = \frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}}$;

if $b = \frac{-3}{5}$, then $a = \frac{4}{3}\left(\frac{-3}{5}\right) = \frac{-4}{5}$

so $\boxed{\vec{u} = -\frac{4}{5}\vec{i} + \frac{-3}{5}\vec{j}}$.

d.) $D_{\vec{u}} f(1,-1) = 4 \rightarrow$

$\vec{\nabla} f(1,-1) \cdot \vec{u} = (3\vec{i} - 4\vec{j}) \cdot (a\vec{i} + b\vec{j})$

$\left.\begin{matrix} = 3a - 4b = 4 \\ \text{and} \quad a^2 + b^2 = 1 \end{matrix}\right\} \begin{matrix} \rightarrow 3a = 4b + 4 \\ \rightarrow \underline{a = \frac{4}{3}b + \frac{4}{3}} \rightarrow \end{matrix}$

(SUB) $\left(\frac{4}{3}b + \frac{4}{3}\right)^2 + b^2 = 1 \rightarrow$

$\frac{16}{9}b^2 + \frac{32}{9}b + \frac{16}{9} + b^2 = 1 \rightarrow$

$16b^2 + 32b + 16 + 9b^2 = 9 \rightarrow$

$25b^2 + 32b + 7 = 0 \rightarrow$

$$b = \frac{-32 \pm \sqrt{(32)^2 - 4(25)(7)}}{2(25)}$$

$$= \frac{-32 \pm \sqrt{324}}{50} = \frac{-32 \pm 18}{50}$$

$= \frac{-14}{50}$ or $\frac{-50}{50} = \frac{-7}{25}$ or -1 ;

if $b = \frac{-7}{25}$, then $a = \frac{4}{3}\left(\frac{-7}{25}\right) + \frac{4}{3}$

$= \frac{-28}{75} + \frac{100}{75} = \frac{72}{75} = \frac{24}{25}$ so

$$\boxed{\vec{u} = \frac{24}{25}\vec{i} + \frac{-7}{25}\vec{j}} \;;$$

if $b = -1$, then $a = \frac{4}{3}(-1) + \frac{4}{3} = 0$ so

$$\vec{u} = (0)\vec{i} + (-1)\vec{j} \rightarrow \boxed{\vec{u} = -\vec{j}}\,.$$

e.) $D_{\vec{u}} f(1,-1) = -3 \rightarrow$

$\vec{\nabla} f(1,-1) \cdot \vec{u} = (3\vec{i} - 4\vec{j}) \cdot (a\vec{i} + b\vec{j})$

$$\left.\begin{aligned} &= 3a - 4b = -3 \\ \text{and } \; & a^2 + b^2 = 1 \end{aligned}\right\} \begin{aligned} &\rightarrow 3a = 4b - 3 \\ &\rightarrow \underbrace{a = \frac{4}{3}b - 1} \end{aligned}$$

$\rightarrow$ (SUB)

$\left(\frac{4}{3}b - 1\right)^2 + b^2 = 1 \rightarrow$

$\frac{16}{9}b^2 - \frac{8}{3}b + \cancel{1} + b^2 = \cancel{1} \rightarrow$

$16b^2 - 24b + 9b^2 = 0 \rightarrow$

$25b^2 - 24b = 0 \rightarrow b(25b - 24) = 0$

$\rightarrow b = 0$ OR $b = \frac{24}{25}$;

if $b=0$, then $a = \frac{4}{3}(0) - 1 = -1$,

so $\vec{u} = (-1)\vec{i} + (0)\vec{j} \rightarrow \boxed{\vec{u} = -\vec{i}}$;

if $b = \frac{24}{25}$, then

$$a = \frac{4}{\cancel{3}}\left(\frac{\overset{8}{\cancel{24}}}{25}\right) - 1$$

$$= \frac{32}{25} - \frac{25}{25} = \frac{7}{25}$$, so

$$\boxed{\vec{u} = \frac{7}{25}\vec{i} + \frac{24}{25}\vec{j}}$$

32.) $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2} \xrightarrow{D}$

$$f_x = \frac{(x^2+y^2)(2x) - (x^2-y^2)(2x)}{(x^2+y^2)^2} \quad \text{and}$$

$$f_x(1,1) = \frac{4-0}{4} = 1 \; ; \xrightarrow{D}$$

$$f_y = \frac{(x^2+y^2)(-2y) - (x^2-y^2)(2y)}{(x^2+y^2)^2} \quad \text{and}$$

$$f_y(1,1) = \frac{-4-0}{4} = -1 \, , \text{ then}$$

$\vec{\nabla} f(1,1) = \overrightarrow{(1,-1)}$; let $\vec{u} = \overrightarrow{(a,b)}$ be unit vector, then

$$D_{\vec{u}} f(1,1) = \vec{\nabla} f(1,1) \cdot \vec{u}$$

$$= \overrightarrow{(1,-1)} \cdot \overrightarrow{(a,b)} = a - b = 0 \rightarrow$$

$$\left.\begin{array}{l} a = b \\ a^2+b^2 = 1 \end{array}\right\} \rightarrow \begin{array}{l} a^2+a^2 = 1 \rightarrow \\ 2a^2 = 1 \rightarrow \boxed{a = \frac{1}{\sqrt{2}},\ b = \frac{-1}{\sqrt{2}}} \end{array}$$

or $\boxed{a = \frac{-1}{\sqrt{2}},\ b = \frac{1}{\sqrt{2}}}$

33.) $f(x,y) = x^2 - 3xy + 4y^2 \xrightarrow{D}$

$f_x = 2x - 3y$, $f_y = -3x + 8y$, then

$\vec{\nabla} f(1,2) = \overrightarrow{(2-6, -3+16)} = \overrightarrow{(-4, 13)}$ and

$|\vec{\nabla} f(1,2)| = \sqrt{(-4)^2 + (13)^2} = \sqrt{185} \approx 13.6$

so $D_{\vec{u}} f(1,2) \neq 14$ since

$-13.6 \leq D_{\vec{u}} f(1,2) \leq +13.6$ (NO)

34.) $T = 2xy - yz \xrightarrow{D}$

$T_x = 2y$, $T_y = 2x - z$, $T_z = -y$, then

$\vec{\nabla} T(1,-1,1) = \overrightarrow{(-2, 2-1, 1)} = \overrightarrow{(-2, 1, 1)}$ and

$|\vec{\nabla} T(1,-1,1)| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6} \approx 2.45$

so $D_{\vec{u}} T(1,-1,1) \neq -3$ since

$-2.45 \leq D_{\vec{u}} T(1,-1,1) \leq +2.45$

35.) If $\vec{A} = \vec{i} + \vec{j}$, then $\vec{u} = \frac{1}{|\vec{A}|}\vec{A} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$

and $D_{\vec{u}} f(1,2) = \vec{\nabla} f(1,2) \cdot \vec{u} = 2\sqrt{2} \Rightarrow$

$\left(f_x(1,2)\vec{i} + f_y(1,2)\vec{j}\right) \cdot \left(\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}\right) = 2\sqrt{2} \Rightarrow$

$f_x(1,2) \cdot \frac{1}{\sqrt{2}} + f_y(1,2) \cdot \frac{1}{\sqrt{2}} = 2\sqrt{2} \Rightarrow$

$\boxed{f_x(1,2) + f_y(1,2) = 4}$;

if $\vec{A} = -2\vec{j}$, then $\vec{u} = \frac{1}{|\vec{A}|}\vec{A} = \frac{1}{2}(-2\vec{j}) = -\vec{j}$

and $D_{\vec{u}} f(1,2) = \vec{\nabla} f(1,2) \cdot \vec{u} = -3 \Rightarrow$

$\left(f_x(1,2)\vec{i} + f_y(1,2)\vec{j}\right) \cdot (-\vec{j}) = -3 \Rightarrow$

$-f_y(1,2) = -3 \Rightarrow \boxed{f_y(1,2) = 3}$; then

$f_x(1,2) + (3) = 4 \Rightarrow \boxed{f_x(1,2) = 1}$;

now the gradient vector is

$\underline{\vec{\nabla} f(1,2)} = f_x(1,2)\vec{i} + f_y(1,2)\vec{j} = \underline{\vec{i} + 3\vec{j}}$;

if $\vec{A} = -\vec{i} - 2\vec{j}$, then $\vec{u} = \frac{1}{|\vec{A}|}\vec{A} = \frac{1}{\sqrt{5}}(-\vec{i} - 2\vec{j})$

$\Rightarrow \vec{u} = \frac{-1}{\sqrt{5}}\vec{i} - \frac{2}{\sqrt{5}}\vec{j}$; then

$$D_{\vec{u}} f(1,2) = \vec{\nabla} f(1,2) \cdot \vec{u}$$
$$= (\vec{i} + 3\vec{j}) \cdot \left(\frac{-1}{\sqrt{5}}\vec{i} - \frac{2}{\sqrt{5}}\vec{j}\right)$$
$$= \frac{-1}{\sqrt{5}} - \frac{6}{\sqrt{5}} = \frac{-7}{\sqrt{5}}$$

Let $P = (a,b,c)$ and $w = f(x,y,z)$.

36.) a.) If $\vec{v} = \vec{i} + \vec{j} - \vec{k}$, then

$$\vec{u} = \frac{1}{|\vec{v}|}\vec{v} = \frac{1}{\sqrt{3}}(\vec{i}+\vec{j}-\vec{k}) = \underline{\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{3}}\vec{k}} ;$$

since $D_{\vec{u}} f(a,b,c) = \vec{\nabla} f(a,b,c) \cdot \vec{u} = 2\sqrt{3} \Rightarrow$

$$\left(f_x(a,b,c)\vec{i} + f_y(a,b,c)\vec{j} + f_z(a,b,c)\vec{k}\right) \cdot \left(\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{3}}\vec{k}\right) = 2\sqrt{3} \Rightarrow$$

$$f_x(a,b,c)\cdot\frac{1}{\sqrt{3}} + f_y(a,b,c)\cdot\frac{1}{\sqrt{3}} - f_z(a,b,c)\cdot\frac{1}{\sqrt{3}} = 2\sqrt{3} \Rightarrow$$

(1) $$\boxed{f_x(a,b,c) + f_y(a,b,c) - f_z(a,b,c) = 6} ;$$

since $D_{\vec{u}} f(a,b,c)$ is the <u>largest</u> value of a directional derivative, $\vec{u}$ and $\vec{\nabla} f(a,b,c)$ point in the same direction, i.e.,

$$\boxed{\vec{\nabla} f(a,b,c) = k\vec{u}}$$

for some constant k. Then

$$f_x(a,b,c)\vec{i} + f_y(a,b,c)\vec{j} + f_z(a,b,c)\vec{k} = \frac{k}{\sqrt{3}}\vec{i} + \frac{k}{\sqrt{3}}\vec{j} - \frac{k}{\sqrt{3}}\vec{k} \Rightarrow$$

$$f_x(a,b,c) = \frac{k}{\sqrt{3}},\ f_y(a,b,c) = \frac{k}{\sqrt{3}},\ f_z(a,b,c) = \frac{-k}{\sqrt{3}} \Rightarrow$$

(use (1) now.) $$\frac{k}{\sqrt{3}} + \frac{k}{\sqrt{3}} - \frac{-k}{\sqrt{3}} = 6 \Rightarrow$$

$$3\frac{k}{\sqrt{3}} = 6 \Rightarrow k = 2\sqrt{3} \Rightarrow$$

$$\boxed{\vec{\nabla} f(a,b,c) = 2\vec{i} + 2\vec{j} - 2\vec{k}} .$$