Section 14.6

1.) $\underbrace{x^2+y^2+z^2}_{f(x,y,z)} = 3$ and $P=(1,1,1)$;

find gradient vector :

$f_x = 2x,\ f_y = 2y,\ f_z = 2z$ so

$\vec{\nabla} f(1,1,1) = f_x(1,1,1)\,\vec{i} + f_y(1,1,1)\,\vec{j} + f_z(1,1,1)\,\vec{k} \Rightarrow$

$\vec{\nabla} f(1,1,1) = 2\vec{i} + 2\vec{j} + 2\vec{k}$; then

a.) tangent plane :

$2(x-1) + 2(y-1) + 2(z-1) = 0 \Rightarrow x+y+z=3$.

b.) normal line :

$$L: \begin{cases} x = 1+2t \\ y = 1+2t \\ z = 1+2t \end{cases}$$

4.) $\underbrace{x^2+2xy-y^2+z^2}_{f(x,y,z)} = 7$ and $P=(1,-1,3)$;

find gradient vector :

$f_x = 2x+2y,\ f_y = 2x-2y,\ f_z = 2z$ so

$\vec{\nabla} f(1,-1,3) = f_x(1,-1,3)\,\vec{i} + f_y(1,-1,3)\,\vec{j} + f_z(1,-1,3)\,\vec{k} \Rightarrow$

$\vec{\nabla} f(1,-1,3) = (0)\vec{i} + (4)\vec{j} + (6)\vec{k} = 4\vec{j} + 6\vec{k}$;

then

a.) tangent plane :

$(0)(x-1) + 4(y+1) + 6(z-3) = 0 \Rightarrow$

$4y + 4 + 6z - 18 = 0 \Rightarrow 4y + 6z = 14 \Rightarrow$

$2y + 3z = 7$.

b.) normal line:

$$L: \begin{cases} x = 1 + (0)t \\ y = -1 + (4)t \\ z = 3 + (6)t \end{cases} \Rightarrow L: \begin{cases} x = 1 \\ y = -1 + 4t \\ z = 3 + 6t \end{cases}$$

9.) $z = \ln(x^2+y^2) \Rightarrow \underbrace{z - \ln(x^2+y^2)}_{f(x,y,z)} = 0$

and $P = (1,0,0)$;

find gradient vector :

$f_x = \dfrac{-2x}{x^2+y^2}$, $f_y = \dfrac{-2y}{x^2+y^2}$, $f_z = 1$ so

$\vec{\nabla} f(1,0,0) = f_x(1,0,0)\,\vec{i} + f_y(1,0,0)\,\vec{j} + f_z(1,0,0)\,\vec{k} \Rightarrow$

$\vec{\nabla} f(1,0,0) = (-2)\vec{i} + (0)\,\vec{j} + (1)\,\vec{k} = -2\vec{i} + \vec{k}$;

then tangent plane is

$(-2)(x-1) + (0)(y-0) + (1)(z-0) = 0 \Rightarrow$

$-2x + 2 + z = 0 \Rightarrow z = 2x - 2$

12.) $z = 4x^2 + y^2 \Rightarrow \underbrace{z - 4x^2 - y^2}_{f(x,y,z)} = 0$

and $P = (1,1,5)$;

find gradient vector :

$f_x = -8x$, $f_y = -2y$, $f_z = 1$ so

$\vec{\nabla} f(1,1,5) = f_x(1,1,5)\,\vec{i} + f_y(1,1,5)\,\vec{j} + f_z(1,1,5)\,\vec{k} \Rightarrow$

$\vec{\nabla} f(1,1,5) = -8\vec{i} - 2\vec{j} + \vec{k}$; then tangent plane

is $-8(x-1) - 2(y-1) + (z-5) = 0 \Rightarrow$

$-8x + 8 - 2y + 2 + z - 5 = 0 \Rightarrow -8x - 2y + z = -5$

14.) $\underbrace{xyz}_{f(x,y,z)} = 1$ and $\underbrace{x^2 + 2y^2 + 3z^2}_{g(x,y,z)} = 6$

and $P = (1,1,1)$; find gradient vectors:

$f_x = yz, \ f_y = xz, \ f_z = xy \Rightarrow$

$\vec{\nabla} f(1,1,1) = (1)\vec{i} + (1)\vec{j} + (1)\vec{k} = \underline{\underline{\vec{i} + \vec{j} + \vec{k}}}$;

$g_x = 2x, \ g_y = 4y, \ g_z = 6z \Rightarrow$

$\vec{\nabla} g(1,1,1) = (2)\vec{i} + (4)\vec{j} + (6)\vec{k} = \underline{\underline{2\vec{i} + 4\vec{j} + 6\vec{k}}}$;

vector $\vec{\nabla} f(1,1,1)$ is $\perp$ to level surface $f(x,y,z) = 1$ at $(1,1,1)$, and vector $\vec{\nabla} g(1,1,1)$ is $\perp$ to level surface $g(x,y,z) = 6$, so vector

$$\boxed{\vec{T} = \vec{\nabla} f(1,1,1) \times \vec{\nabla} g(1,1,1)}$$

is tangent to both curves (curve of intersection) at $\underline{\underline{P = (1,1,1)}}$; then

$$\vec{T} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{vmatrix} = (6-4)\vec{i} - (6-2)\vec{j} + (4-2)\vec{k}$$
$$= 2\vec{i} - 4\vec{j} + 2\vec{k} ,$$

so tangent line is

$$L: \begin{cases} x = 1 + (2)t \\ y = 1 + (-4)t \\ z = 1 + (2)t \end{cases} \Rightarrow L: \begin{cases} x = 1 + 2t \\ y = 1 - 4t \\ z = 1 + 2t \end{cases}$$

15.) $\underbrace{x^2+2y+2z}_{f(x,y,z)}=4$ and $\underbrace{y}_{g(x,y,z)}=1$

and $P=(1,1,\frac{1}{2})$; find gradient vectors:

$f_x=2x,\ f_y=2,\ f_z=2$ so

$\vec{\nabla}f(1,1,\frac{1}{2})=2\vec{i}+2\vec{j}+2\vec{k}$;

$g_x=0,\ g_y=1,\ g_z=0$ so

$\vec{\nabla}g(1,1,\frac{1}{2})=(0)\vec{i}+(1)\vec{j}+(0)\vec{k}=\vec{j}$; then

$$\vec{T}=\vec{\nabla}f(1,1,\tfrac{1}{2})\times\vec{\nabla}g(1,1,\tfrac{1}{2})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 2 & 2\\ 0 & 1 & 0\end{vmatrix}$$

$=(0-2)\vec{i}-(0-0)\vec{j}+(2-0)\vec{k}=-2\vec{i}+2\vec{k}$;

so tangent line is

$$L:\begin{cases}x=1+(-2)t\\ y=1+(0)t\\ z=\frac{1}{2}+(2)t\end{cases}\Rightarrow L:\begin{cases}x=1-2t\\ y=1\\ z=\frac{1}{2}+2t.\end{cases}$$

20.) $f(x,y,z)=e^x\cos yz$, $P=(0,0,0)$,

$ds=0.1$, and $\vec{A}=2\vec{i}+2\vec{j}-2\vec{k}$; then

$\vec{u}=\frac{1}{|\vec{A}|}\vec{A}=\frac{1}{2\sqrt{3}}(2\vec{i}+2\vec{j}-2\vec{k})\Rightarrow$

$\vec{u}=\frac{1}{\sqrt{3}}\vec{i}+\frac{1}{\sqrt{3}}\vec{j}-\frac{1}{\sqrt{3}}\vec{k}$; find

gradient vector :

$f_x = e^x \cos yz$, $f_y = -ze^x \sin yz$, $f_z = -ye^x \sin yz$

so $\vec{\nabla} f(0,0,0) = (1)\vec{i} + (0)\vec{j} + (0)\vec{k} = \vec{i}$; then

the <u>differential</u> is

$$df = \left(D_{\vec{u}} f(0,0,0)\right) \cdot ds$$
$$= \left(\vec{\nabla} f(0,0,0) \cdot \vec{u}\right) \cdot ds$$
$$= \left(\vec{i} \cdot \left(\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{3}}\vec{k}\right)\right) \cdot (0.1)$$
$$= \frac{1}{\sqrt{3}}(0.1) \approx 0.0577$$

21.) $g(x,y,z) = x + x\cos z - y\sin z + y$,
$P = (2,-1,0)$, $ds = 0.2$, move toward $Q = (0,1,2)$ so
$\vec{PQ} = \overrightarrow{(-2,2,2)}$; then $\vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|} \Rightarrow$

$\vec{u} = \frac{1}{2\sqrt{3}}\overrightarrow{(-2,2,2)} = \overrightarrow{\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$;

find gradient vector :

$g_x = 1 + \cos z$, $g_y = -\sin z + 1$,
$g_z = -x\sin z - y\cos z$ so

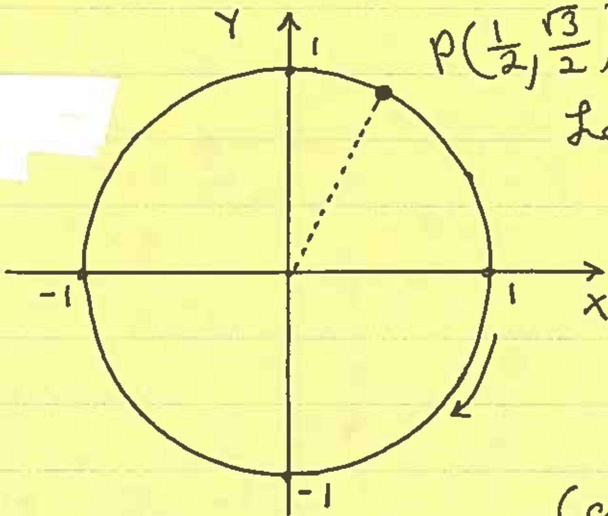
$\vec{\nabla} g(2,-1,0) = (2)\vec{i} + (1)\vec{j} + (1)\vec{k} = 2\vec{i} + \vec{j} + \vec{k}$; then the <u>differential</u> is

$$df = \left(D_{\vec{u}}\, g(2,-1,0)\right) \cdot ds$$
$$= \left(\vec{\nabla} g(2,-1,0) \cdot \vec{u}\right) \cdot ds$$

$$= \left((2\vec{i} + \vec{j} + \vec{k}) \cdot \left(-\tfrac{1}{\sqrt{3}}\vec{i} + \tfrac{1}{\sqrt{3}}\vec{j} + \tfrac{1}{\sqrt{3}}\vec{k} \right) \right) \cdot (0.2)$$

$$= \left(\tfrac{-2}{\sqrt{3}} + \tfrac{1}{\sqrt{3}} + \tfrac{1}{\sqrt{3}} \right)(0.2) = (0)(0.2) = 0 .$$

23.)



$P\left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right)$ for $t = \tfrac{5}{6}\pi$

Let $\begin{cases} x = \cos 2t \\ y = -\sin 2t \end{cases}$

for $t : 0 \to \pi$

traces circle of radius 1 since

(clockwise)

$$x^2 + y^2 = (\cos 2t)^2 + (-\sin 2t)^2$$

$$= \cos^2 2t + \sin^2 2t = 1$$; speed is

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(-2\sin 2t)^2 + (-2\cos 2t)^2}$$

$$= \sqrt{4(\sin^2 2t + \cos^2 2t)} = \sqrt{4(1)} = 2 ;$$

temperature $T = x \sin 2y \rightarrow$

$$T = \cos 2t \cdot \sin(-2\sin 2t) \xrightarrow{D}$$

$$\frac{dT}{dt} = \cos 2t \cdot \cos(-2\sin 2t) \cdot (-4\cos 2t)$$

$$+ (-2\sin 2t) \cdot \sin(-2\sin 2t)$$ (let $t = \tfrac{5}{6}\pi$)

$$= \left(\tfrac{1}{2}\right)\cos\left(-2 \cdot \tfrac{-\sqrt{3}}{2}\right)\left(-4 \cdot \left(\tfrac{1}{2}\right)\right)$$

$$- 2\left(\tfrac{-\sqrt{3}}{2}\right) \cdot \sin\left(-2 \cdot \left(-\tfrac{\sqrt{3}}{2}\right)\right)$$

$= -\cos\sqrt{3} + \sqrt{3}\sin\sqrt{3} \approx 1.87$

a.) $\dfrac{dT}{ds} = \dfrac{\frac{dT}{dt}}{\frac{ds}{dt}} = \dfrac{\sqrt{3}\sin\sqrt{3} - \cos\sqrt{3}}{2} \approx 0.935 \ \dfrac{°C}{m}$

b.) $\dfrac{dT}{dt} = \sqrt{3}\sin\sqrt{3} - \cos\sqrt{3} \approx 1.87 \ \dfrac{°C}{sec}.$

26.) b.) $f(x,y) = (x+y+2)^2$, pt. $(1,2)$

$\rightarrow f(1,2) = (5)^2 = 25$ and

$f_x = 2(x+y+2)$, $f_y = 2(x+y+2)$ so

$f_x(1,2) = 10$, $f_y(1,2) = 10$; lin. at $(1,2)$ is

$L(x,y) = 25 + 10(x-1) + 10(y-2)$

$= 25 + 10x - 10 + 10y - 20$

$= 10x + 10y - 5$

28.) a.) $f(x,y) = x^3y^4$, pt. $(1,1)$

$\rightarrow f(1,1) = 1$ and $f_x = 3x^2y^4$, $f_y = 4x^3y^3$ so

$f_x(1,1) = 3$, $f_y(1,1) = 4$; lin. at $(1,1)$ is

$L(x,y) = 1 + 3(x-1) + 4(y-1)$

$= 1 + 3x - 3 + 4y - 4$

$= 3x + 4y - 6$

30.) b.) $f(x,y) = e^{2y-x}$, pt. $(1,2)$

$\rightarrow f(1,2) = e^{4-1} = e^3$ and

$f_x = -e^{2y-x}$, $f_y = 2e^{2y-x}$ so

$f_x(1,2) = -e^3$, $f_y(1,2) = 2e^3$; lin. at $(1,2)$

is $L(x,y) = e^3 - e^3(x-1) + 2e^3(y-2)$

$= \cancel{e^3} - e^3 x + \cancel{e^3} + 2e^3 y - 4e^3$

$= 2e^3 y - e^3 x - 2e^3$

52.) a.) $f(x,y) = x^2 y + x^2$, pt. $(1,0)$;

$f_x = 2xy + 2x$, $f_x(1,0) = 2(1)(0) + 2(1) = 2$,

$f_y = x^2$, $f_y(1,0) = 1^2 = 1$; SLOPE (2) in x-direction is larger than SLOPE (1) in y-direction, so f is more sensitive to changes in x near pt. $(1,0)$

b.) $df = f_x(1,0)\,dx + f_y(1,0)\,dy$

$= (2)\,dx + (1)\,dy = 0 \rightarrow$

$2\,dx = -dy \rightarrow \dfrac{dx}{dy} = \dfrac{-1}{2}$

56.) $f(x,y) = x^2+y^2$, curve

$\vec{r}(t) = (\cos t + t\sin t)\vec{i}$
$+ (\sin t - t\cos t)\vec{j}$ has

tangent vector

$\vec{r}\,'(t) = (\cancel{-\sin t} + t\cos t + \cancel{\sin t})\vec{i}$
$+ (\cancel{\cos t} - (t \cdot -\sin t + (1)\cancel{\cos t}))\vec{j}$
$= t\cos t\,\vec{i} + t\sin t\,\vec{j}$,

$|\vec{r}\,'(t)| = \sqrt{(t\cos t)^2 + (t\sin t)^2}$
$= \sqrt{t^2(\cos^2 t + \sin^2 t)} = \sqrt{t^2(1)} = t$, so

unit tangent vector is

$\vec{u} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{t\cos t\,\vec{i} + t\sin t\,\vec{j}}{t} \rightarrow$

$\boxed{\vec{u} = (\cos t)\vec{i} + (\sin t)\vec{j}}$;

$\vec{\nabla} f = \overrightarrow{(f_x, f_y)} = \overrightarrow{(2x, 2y)} \rightarrow$

$\boxed{\vec{\nabla} f = \overrightarrow{(2(\cos t + t\sin t), 2(\sin t - t\cos t))}}$;

$$\begin{cases} x = \cos t + t\sin t \\ y = \sin t - t\cos t \end{cases}, \quad t > 0$$

so directional derivative of f in direction of $\vec{r}'(t)$ at pt (x,y) is

$$D_{\vec{u}}f(x,y) = \vec{\nabla} f(x,y) \cdot \vec{u}$$

$$= 2(\cos t + t\sin t)\cdot \cos t + 2(\sin t - t\cos t)\cdot \sin t$$

$$= 2\cos^2 t + 2t\cancel{\sin t}\cos t + 2\sin^2 t - 2t\cancel{\sin t}\cos t$$

$$= 2(\cos^2 t + \sin^2 t)$$

$$= 2(1)$$

$$= 2$$